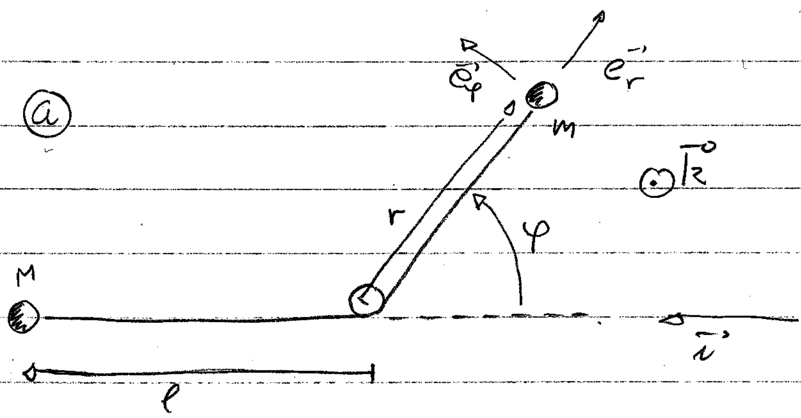
Examen Mecánica Newtoniana 28/07/11

① ⓐ



$r + \ell = c^{te}$

$\dot r = -\dot\ell$

$\ddot r = -\ddot\ell$

$$\vec{v_m} = \dot r \vec{e_r} + r\dot\varphi \vec{e_\varphi} \quad ; \quad \vec{a_m} = (\ddot r - r\dot\varphi^2)\vec{e_r} + (r\ddot\varphi + 2\dot r\dot\varphi)\vec{e_\varphi}$$

$$\vec{L_m} = \vec{r} \times \vec{v_m} = r\vec{e_r} \times (\dot r \vec{e_r} + r\dot\varphi\vec{e_\varphi}) = r^2\dot\varphi\,\vec{k}$$

$$\vec{L_M} \equiv \emptyset$$

$$\vec{L} = \vec{L_m} + \vec{L_M}$$

$$m\vec{a_m} = -T\vec{e_r} \Rightarrow \begin{cases} m(\ddot r - r\dot\varphi^2) = -T \\ m(r\ddot\varphi + 2\dot r\dot\varphi) = \emptyset \end{cases}$$

$$\rightarrow \dot{\vec{L}} = c^{te} \Rightarrow r^2\dot\varphi = \mathbb{C}$$

$$\dot\varphi^2 = \frac{\mathbb{C}^2}{r^4}$$

$$\begin{cases} m\left(\ddot r - \dfrac{\mathbb{C}^2}{r^3}\right) = -T \\ M\ddot\ell\,\vec{i} = -T\vec{i} \Rightarrow -M\ddot r = -T \end{cases}$$

$$\rightarrow m\ddot r - \frac{m\mathbb{C}^2}{r^3} = -M\ddot r \Rightarrow \boxed{\ddot r - \frac{m\mathbb{C}^2}{(m+M)r^3} = \emptyset}$$

$$\boxed{\dot\varphi = \mathbb{C}/r^2}$$

(b) $u(\varphi)=\dfrac{1}{r} \overset{\text{Binet}}{\Longrightarrow} u'\dot\varphi = -\dfrac{1}{r^2}\dot r \Rightarrow u' = \dfrac{-\dot r}{r^2\dot\varphi} = \dfrac{-\dot r}{C}$

$$u''\dot\varphi = -\frac{\ddot r}{C} \Rightarrow u'' = \frac{-\ddot r}{r^2\dot\varphi^2} = -\frac{\ddot r\, r^2}{C^2}$$

De la ec. de movimiento $\qquad u''u^2 = -\dfrac{\ddot r}{C^2}$

$$-u''u^2C^2 - \frac{m}{m+M}C^2u^3 = 0$$

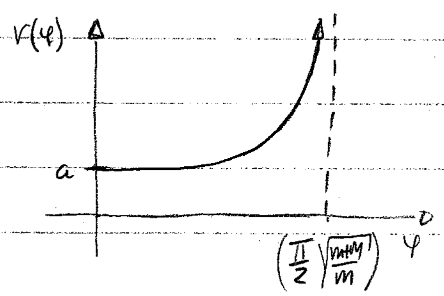
$$u'' + \frac{m}{m+M}u = 0$$

$u(\varphi) = A\cos(k\varphi+\beta)$; $u(0)=\dfrac{1}{a}$

$k=\sqrt{\dfrac{m}{m+M}}$ $\qquad u'(0)=0$

$A=\dfrac{1}{a}$, $\beta=0$.

$u(\varphi) = \dfrac{1}{a}\cos\left(\sqrt{\dfrac{m}{m+M}}\,\varphi\right)$ ← trayectoria de la partícula m



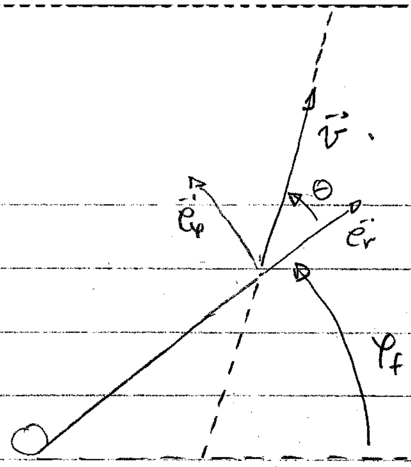
$$\frac{\pi}{2}\sqrt{\frac{m+M}{m}} > \pi$$

$$\frac{m+M}{m} > 4 \Rightarrow \boxed{M > 3m}$$

$$T = M\ddot r \underset{\text{ec. mov.}}{=} \frac{Mm\, C^2}{m+M\; r^3} = \left(\frac{mMa^2v_0^2}{m+M}\right)\frac{1}{r^3}$$

$$C^2 = (a v_0)^2$$

① ⓒ



$$\vec v = \dot r\,\vec e_r + r\dot\varphi\,\vec e_\varphi = -\mathbb{C}u'\,\vec e_r + \mathbb{C}u\,\vec e_\varphi$$

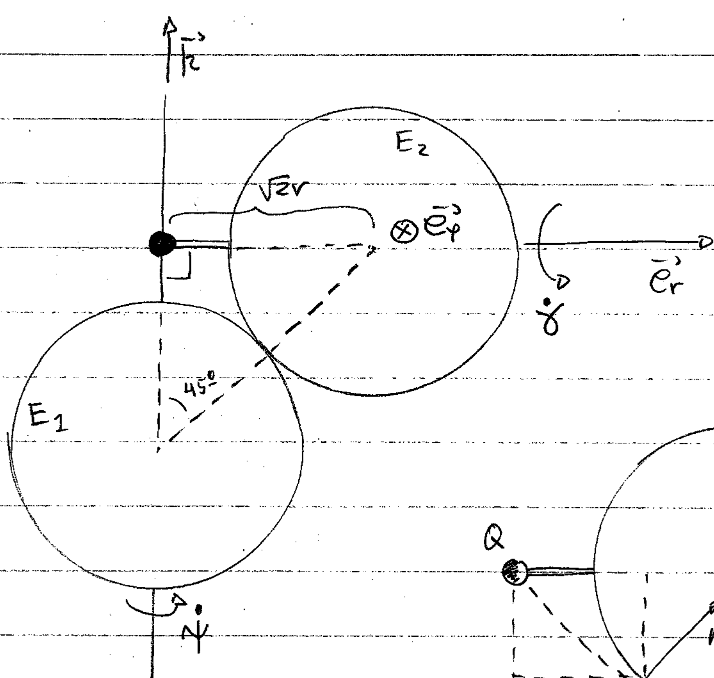
$$\vec v(\varphi_f) = -\mathbb{C}\left(-\frac{\sqrt{\frac{m}{m+M}}}{a}\,\mathrm{sen}\left(\sqrt{}\,\varphi_f\right)\right)\vec e_r + $$

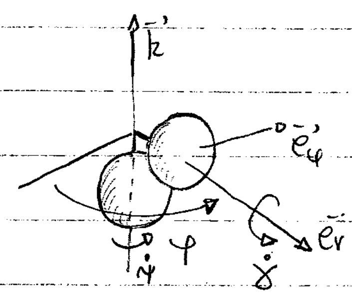
$$+\ \mathbb{C}\left(\frac{1}{a}\cos\left(\sqrt{}\,\varphi_f\right)\right)\vec e_\varphi$$

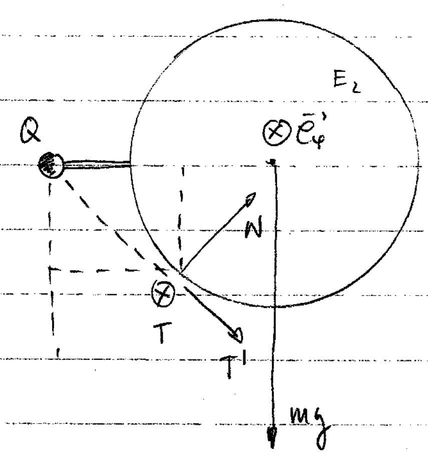
$$\mathrm{tg}\,\theta = \frac{\cos\left(\sqrt{}\,\varphi_f\right)}{\mathrm{sen}\left(\sqrt{}\,\varphi_f\right)} = \mathrm{ctg}\left(\sqrt{}\,\varphi_f\right)$$

$$\boxed{\theta + \varphi_f = \mathrm{Arcctg}\left(\sqrt{\frac{m}{m+M}}\,\varphi_f\right) + \varphi_f}$$

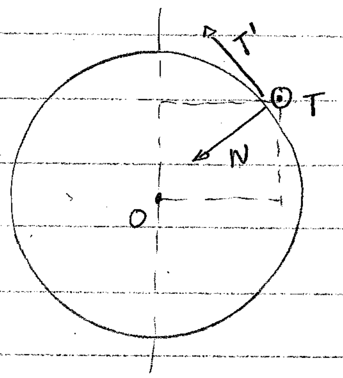
②







$$\vec M_Q^{ext} = mg\sqrt{2}r\,\vec e_\varphi - N r\,\vec e_\varphi + T\frac{\sqrt2}{2}r\,\vec e_r + T\frac{\sqrt2}{2}r\,\vec k$$



$$\vec{M}_O^{ext} = -T' r\, \vec{e}_\varphi + \cancel{M^{rest}}\, \vec{e}_\varphi + \cancel{M^{rest'}}\, \vec{e}_r + T\frac{\sqrt{2}}{2} r\, \vec{e}_r - T\frac{\sqrt{2}}{2} r\, \vec{k}$$

$$\left.\begin{array}{l} \mathbb{I}_O = \frac{2}{5} m r^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\ \vec{\omega}_1 = \dot{\varphi}\, \vec{k} \end{array}\right\} \Rightarrow \mathbb{I}_O \vec{\omega}_1 = \frac{2 m r^2 \dot{\varphi}}{5} \vec{k}$$

$$\left(\frac{d\, \mathbb{I}_O \vec{\omega}_1}{dt}\right) \cdot \vec{k} = \frac{2}{5} m r^2 \ddot{\varphi} = \vec{M}_O^{ext} \cdot \vec{k} = -T\frac{\sqrt{2}}{2} r$$

$$\boxed{\frac{2}{5} m r^2 \ddot{\varphi} = -\frac{\sqrt{2}}{2} r T} \quad (i)$$

E_2:

$$\left.\begin{array}{l} \underset{(e_r, e_\varphi, k)}{\mathbb{I}_Q} = \mathbb{I}_G + \begin{pmatrix} 0 & & \\ & 2mr^2 & \\ & & 2mr^2 \end{pmatrix} \\ \vec{\omega}_2 = \dot{\gamma}\, \vec{e}_r + \dot{\varphi}\, \vec{k} \end{array}\right\} \quad \frac{d}{dt}\left(\mathbb{I}_Q \vec{\omega}_2\right) = \frac{d}{dt}\left(\frac{2}{5} m r^2 \dot{\gamma}\, \vec{e}_r + \frac{12}{5} m r^2 \dot{\varphi}\, \vec{k}\right) =$$

$$= \frac{2}{5} m r^2 \left(\ddot{\gamma}\, \vec{e}_r + \dot{\gamma}\dot{\varphi}\, \vec{e}_\varphi + 6\ddot{\varphi}\, \vec{k}\right) =$$

$$= m g \sqrt{2} r\, \vec{e}_\varphi - N r\, \vec{e}_\varphi + T\frac{\sqrt{2}}{2} r\, \vec{e}_r + T\frac{\sqrt{2}}{2} r\, \vec{k}$$

$$\boxed{\frac{2}{5} m r^2 \ddot{\gamma} = \frac{\sqrt{2}}{2} r T} \quad (ii) \qquad (i) + (ii) \Rightarrow \quad \ddot{\gamma} = -\ddot{\varphi}, \quad \dot{\gamma} = \Omega_0 - \dot{\varphi}$$

$$\boxed{\frac{2}{5} m r^2 (6\ddot{\varphi}) = \frac{\sqrt{2}}{2} r T} \quad (iii) \qquad (ii) + (iii) \rightarrow \quad \ddot{\gamma} = 6\ddot{\varphi}, \quad \dot{\gamma} = 6\dot{\varphi}$$

La vel. angular $\dot{\varphi}$ se reduce y $\dot{\gamma}$ aumenta (al igual que $\dot{\varphi}$) hasta que llega a régimen el movimiento en la situación de rodadura sin deslizamiento. Sea C: punto de contacto entre las esferas.

$$\vec{v}_O = \frac{\sqrt{2}}{2} r \dot{\varphi}\, \vec{e}_\varphi \underset{R.S.D}{=} \sqrt{2} r \dot{\varphi}\, \vec{e}_\varphi + \vec{\omega}_2 \times \left(-\frac{\sqrt{2}}{2} r\, \vec{e}_r - \frac{\sqrt{2}}{2} r\, \vec{k}\right) =$$

$$= \sqrt{2} r \dot{\varphi}\, \vec{e}_\varphi + \frac{\sqrt{2}}{2} r \dot{\gamma}\, \vec{e}_\varphi - \frac{\sqrt{2}}{2} r \dot{\varphi}\, \vec{e}_\varphi \Rightarrow \boxed{\dot{\varphi} = \dot{\gamma} + \dot{\varphi}}$$

Entonces, velocidades angulares de régimen:

$$\left.\begin{array}{l} \dot{\gamma} = \Omega_0 - \dot{\psi} \\ \dot{\gamma} = 6\dot{\varphi} \\ \dot{\psi} = \dot{\gamma} + \dot{\varphi} \end{array}\right\} \Rightarrow \begin{array}{l} 6\dot{\varphi} = \Omega_0 - \dot{\psi} \\ 7\dot{\varphi} = \dot{\psi} \end{array} \quad \begin{array}{l} 7(\Omega_0 - \dot{\psi}) = 6\dot{\psi} \\ 7\Omega_0 = 13\dot{\psi} \end{array}$$

$$\boxed{\dot{\psi} = \frac{7}{13}\Omega_0}$$

(b) $E_{mec} = T_{E_1} + T_{E_2}$

$$T_{E_1} = \frac{1}{2}\vec{\omega}_1 \, \mathbb{I}_G \, \vec{\omega}_1 = \frac{1}{2}\dot{\psi}\vec{k}\, \mathbb{I}_{G_1} \cdot \dot{\psi}\vec{k} = \frac{1}{5}mr^2\dot{\psi}^2$$

$$\Delta T_{E_1} = \frac{1}{5}mr^2\left(\frac{49}{169}\Omega_0^2 - \Omega_0^2\right) = -\frac{24}{169}mr^2\Omega_0^2$$

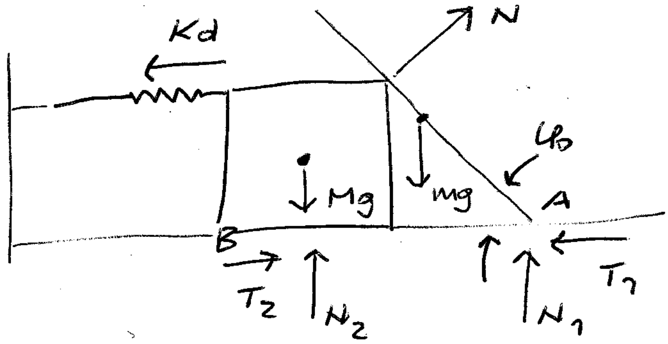
$$T_{E_2} = \frac{1}{2}\vec{\omega}_2 \, \mathbb{I}_Q \, \vec{\omega}_2 = \frac{1}{2}(\dot{\gamma}\vec{e}_r + \dot{\varphi}\vec{k})\begin{pmatrix} \frac{2}{5}mr^2 & & \\ & \frac{12}{5}mr^2 & \\ & & \frac{12}{5}mr^2 \end{pmatrix}(\dot{\gamma}\vec{e}_r + \dot{\varphi}\vec{k}) =$$

$$= \frac{mr^2}{5}\left(\dot{\gamma}^2 + 6\dot{\varphi}^2\right) =$$

$$= \frac{mr^2}{5}\left(36\dot{\varphi}^2 + 6\dot{\varphi}^2\right) = \frac{42mr^2}{5}\dot{\varphi}^2$$

$$\Delta T_{E_2} = \frac{42}{5}mr^2\frac{\dot{\psi}^2}{49} = \frac{42mr^2\Omega_0^2}{5 \cdot 169}$$

$$\boxed{\Delta E = \frac{-78}{845}mr^2\Omega_0^2}$$



(a) 2$^{\text{da}}$ Cardinal a la barra desde A:

$$N\,a/\operatorname{sen}\varphi_0 = mg\,L/2\,\cos\varphi_0 \quad (1)$$

1$^{\text{era}}$ Cardinal a la barra:

$$N\cos\varphi_0 + N_1 = mg \quad (2)$$

$$N\operatorname{sen}\varphi_0 = T_1 \quad (3)$$

1$^{\text{era}}$ Cardinal a la placa: $N_2 = Mg + N\cos\varphi_0 \quad (4)$

$$T_2 = Kd + N\operatorname{sen}\varphi_0 \quad (5)$$

Condiciones de equilibrio:

$N_1 \geq 0$: a partir de (1) $N = \dfrac{mgL}{2a}\operatorname{sen}\varphi_0\cos\varphi_0$

(no despren. de la barra) sustituyendo en (2): $N_1 = mg\left(1 - \dfrac{L}{2a}\operatorname{sen}\varphi_0\cos^2\varphi_0\right) \geq 0$

$$\boxed{a \geq \frac{L}{2}\operatorname{sen}\varphi_0\cos^2\varphi_0}$$

no deslizamiento de la barra: $|T_1| \leq f_E N_1$

usando (3) y (1): $T_1 = \dfrac{\cancel{mg}L}{2a}\operatorname{sen}^2\varphi_0\cos\varphi_0 \leq f_E\,\cancel{mg}\left(1 - \dfrac{L}{2a}\operatorname{sen}\varphi_0\cos^2\varphi_0\right)$

$$\boxed{f_E \geq \frac{\frac{L}{2a}\operatorname{sen}^2\varphi_0\cos\varphi_0}{1 - \frac{L}{2a}\operatorname{sen}\varphi_0\cos^2\varphi_0}}$$

no deslizamiento de la placa: $|T_2| \leq f_E N_2$

de (5) y (1): $T_2 = Kd + \dfrac{mgL}{2a}\operatorname{sen}^2\varphi_0\cos\varphi_0$

de (4) y (1): $N_2 = Mg + \dfrac{mgL}{2a}\operatorname{sen}\varphi_0\cos^2\varphi_0$

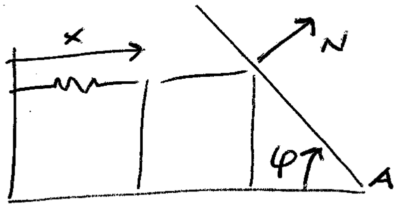
$$\boxed{f_E \geq \frac{Kd + \frac{mgL}{2a}\operatorname{sen}^2\varphi_0\cos\varphi_0}{Mg + \frac{mgL}{2a}\operatorname{sen}\varphi_0\cos^2\varphi_0}}$$

no vuelco de la placa: considerando la 2$^{\text{da}}$ cardinal desde B:

$$aKd - \frac{a}{2}Mg - aN\cos\varphi_0 + aN\operatorname{sen}\varphi_0 \leq 0$$

$$\boxed{Kd \leq \frac{Mg}{2} + \frac{mgL}{2a}\operatorname{sen}\varphi_0\cos\varphi_0(\cos\varphi_0 - \operatorname{sen}\varphi_0)}$$

(b)



La energía del sistema se conserva:

$$T + U = cte.$$

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I_A \dot{\varphi}^2$$

$$U = \frac{1}{2} K x^2 + mg\frac{L}{2}\operatorname{sen}\varphi$$

La distancia entre A y la pared izq. es cte:

$$\boxed{x + \frac{a\cos\varphi}{\operatorname{sen}\varphi} = d + \frac{a\cos\varphi_0}{\operatorname{sen}\varphi_0}}$$

$$\Rightarrow \dot{x} = \frac{a\dot{\varphi}}{\operatorname{sen}^2\varphi}$$

$$\boxed{E = \frac{1}{2} M\left(\frac{a\dot{\varphi}}{\operatorname{sen}^2\varphi}\right)^2 + \frac{1}{2}\left(\frac{mL^2}{3}\right)\dot{\varphi}^2 + \frac{1}{2}K\left(d + \frac{a\cos\varphi_0}{\operatorname{sen}\varphi_0} - \frac{a\cos\varphi}{\operatorname{sen}\varphi}\right)^2 + mg\frac{L}{2}\operatorname{sen}\varphi}$$

(c) 2da Cardinal a la barra desde A:

$$I_A\ddot{\varphi} = \frac{Na}{\operatorname{sen}\varphi} - mg\frac{L}{2}\cos\varphi$$

$$\frac{aN(0)}{\underbrace{\operatorname{sen}\varphi_0}_{>0}} = I_A\ddot{\varphi}(0) + mg\frac{L}{2}\cos\varphi_0 \geq 0$$ para que no se desprenda en el instante inicial

derivando la cons. de E: $$0 = \frac{Ma^2}{\operatorname{sen}^4\varphi_0}\ddot{\varphi}(0) + \frac{mL^2}{3}\ddot{\varphi}(0) + \cdots$$

$$\cdots + K\left(d + \cancel{\frac{a\cos\varphi_0}{\operatorname{sen}\varphi_0}} - \cancel{\frac{a\cos\varphi_0}{\operatorname{sen}\varphi_0}}\right)\frac{a}{\operatorname{sen}^2\varphi_0} + mg\frac{L}{2}\cos\varphi_0$$

$$-\ddot{\varphi}(0) = \left(\frac{mL^2}{3} + \frac{Ma^2}{\operatorname{sen}^4\varphi_0}\right)^{-1}\left(\frac{Kda}{\operatorname{sen}^2\varphi_0} + mg\frac{L}{2}\cos\varphi_0\right)$$

$$\Rightarrow \boxed{-\frac{mL^2}{3}\left(\frac{mL^2}{3} + \frac{Ma^2}{\operatorname{sen}^4\varphi_0}\right)^{-1}\left(\frac{Kda}{\operatorname{sen}^2\varphi_0} + mg\frac{L}{2}\cos\varphi_0\right) + mg\frac{L}{2}\cos\varphi_0 \geq 0}$$