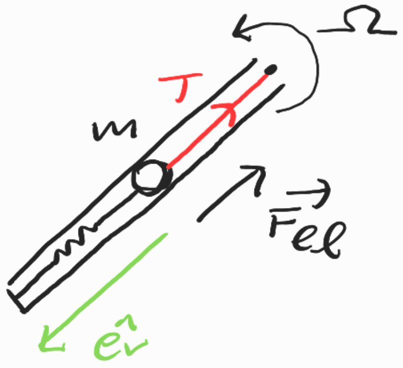
Mecánica Newtoniana, Primer parcial (25/09/23)

Ej 7) a)




2da Ley de Newton a la masa:

$$\vec F = m\vec a \;;$$

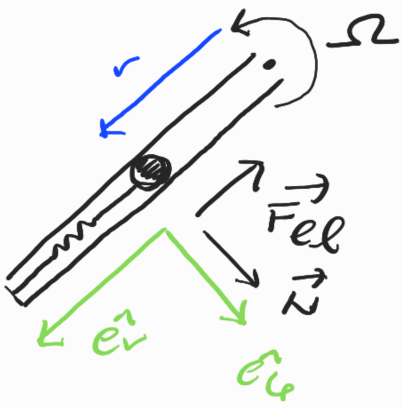
$$K\frac{R}{2} + T = m\frac{R}{2}\Omega^2 \;;$$

$$T = \frac{R}{2}\left(m\Omega^2 - K\right)$$

$0 \le T \le T_{max} = KR$ (el hilo sólo puede tirar y soporta una tensión máxima $T_{max}$)

$$0 \le \frac{R}{2}\left(m\Omega^2 - K\right) \le KR \;; \quad \boxed{\underbrace{\frac{K}{m}}_{\Omega^2_{min}} \le \Omega^2 \le \underbrace{3\frac{K}{m}}_{\Omega^2_{max}}}$$

b) $\Omega^2 = 4\frac{K}{m} > \Omega^2_{max}$ ; el hilo se rompe desde el comienzo:




$$\vec F_{el} = K\left(R - r - R\right)\hat e_r = -Kr\,\hat e_r$$

2da Ley de Newton según la dirección radial:

$$-Kr = m\left(\ddot r - r\Omega^2\right) \;;$$

$$\ddot r + \underbrace{\left(\frac{K}{m} - \Omega^2\right)}_{= -3\frac{K}{m}} r = 0 \;; \quad \boxed{\ddot r - \underbrace{3\frac{K}{m}}_{\omega^2} r = 0}$$

$\vec a_1 = \ddot r\,\hat e_r$ ; $\vec a_1(0) = 3\frac{K}{m}\frac{R}{2}\,\hat e_r$ (la partícula tiende a alejarse del eje de giro)

c) ensayo una solución de la forma: $r(t) = Ae^{\lambda t}$

$\xrightarrow{\text{Rest. en ec. mov}}$ $\lambda^2 \cancel{Ae^{\lambda t}} - \omega^2 \cancel{Ae^{\lambda t}} = 0$ : $\lambda^2 - \omega^2 = 0$ polinomio característico

$\lambda = \pm\omega$ : $r(t) = A_+ e^{+\omega t} + A_- e^{-\omega t}$

$r(0) = R/2$ : $A_+ + A_- = R/2$

$\dot r(0) = 0$ : $\omega(A_+ - A_-) = 0$

$A_+ = A_- = R/4$ :

$$r(t) = \frac{R}{2}\left(\frac{e^{\omega t} + e^{-\omega t}}{2}\right) = \boxed{\frac{R}{2}\,\mathrm{ch}(\omega t)}$$

(crece con t, se aleja del eje de giro ✓)

d) i) $\vec v = \dot r \hat{e_r} + r\Omega \hat{e_\varphi}$, necesitamos $\dot r\,(r = 3/4R)$ ;

preintegro la ec. de movimiento: $\dot r^2 - \omega^2 r^2 = C = 0 - \omega^2 \left(\frac{R}{2}\right)^2$ :

$\dot r^2 = \omega^2 (r^2 - R^2/4)$ ; $\dot r^2(3/4R) = 5\omega^2 R^2/16$

$$\Rightarrow \vec v|_{r=3R/4} = \sqrt5\,\omega R/4\,\hat{e_r} + \Omega\,3R/4\,\hat{e_\varphi}$$

ii) 2$^{da}$ Ley de Newton según la dirección tangencial:

$$N = m\,2\Omega \dot r$$

$P_N = \vec N \cdot \vec v$ , $\vec N = m2\Omega\dot r\,\hat{e_\varphi}$, $\vec v = \dot r\hat{e_r} + r\Omega\hat{e_\varphi}$ | $P_N = 2m\Omega^2 r\dot r$

$$W_N = \int_0^{t^*} P_N\,dt = 2m\Omega^2 \int_0^{t^*} r\dot r\,dt = m\Omega^2 r^2\Big|_0^{t^*}$$

instante en que $r = 3/4R$

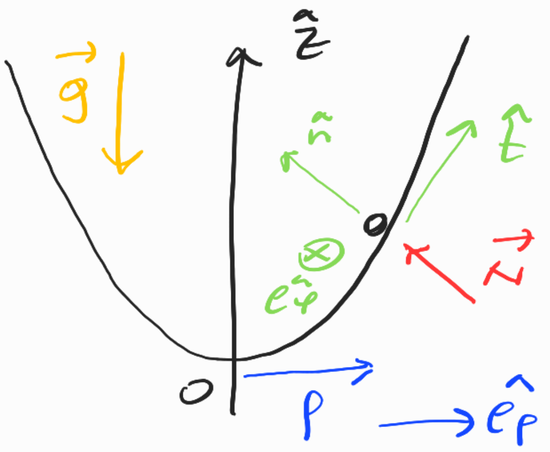
$\frac{d}{dt}\left(\frac{1}{2}r^2\right)$

$$= m\Omega^2\left[\left(\frac{3}{4}R\right)^2 - \left(\frac{R}{2}\right)^2\right] = \frac{5m\Omega^2R^2}{16} = \boxed{\frac{5}{4}KR^2}$$

(Idem: $W_N = \Delta E$, $E_0 = \frac{1}{2}m\vec v_0^{\,2} + U_{el}(R/2)$

$E_f = \frac{1}{2}m\vec v_f^{\,2} + U_{el}(3R/4)$ )

Ej 2) a)




$z = \alpha\rho^2$

$\vec{r} = \rho\hat{e_\rho} + z\hat{z}$

$\vec{v} = \dot{\rho}\hat{e_\rho} + \rho\dot{\varphi}\hat{e_\varphi} + \dot{z}\hat{z}$

$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{e_\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{e_\varphi} + \ddot{z}\hat{z}$

i) $L_z = \vec{L_o}\cdot\hat{z}$ , $\vec{L_o} = \vec{r}\times\vec{p}$ , $\vec{p} = m\vec{v}$

$$\Rightarrow L_z = (\rho\hat{e_\rho} + z\hat{z})\times m(\dot{\rho}\hat{e_\rho} + \rho\dot{\varphi}\hat{e_\varphi} + \dot{z}\hat{z}).\hat{z} = \boxed{m\rho^2\dot{\varphi}}$$

$$\dot{L_z} = 2m\rho\dot{\rho}\dot{\varphi} + m\rho^2\ddot{\varphi} = m\rho\underbrace{(\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})}_{= \vec{a}\cdot\hat{e_\varphi} = 0}$$

( No hay fuerzas según $\hat{e_\varphi}$ )

ii) $\dot{E} = \mathcal{P}^{(vel)}$

La única fuerza no conservativa actuando sobre la partícula es $\vec{N}$

pero : $\mathcal{P}_N = \vec{N}\cdot\vec{v} = (N\hat{n})\cdot\underbrace{(v_t\hat{t} + v_\varphi\hat{e_\varphi})}_{\vec{v}\text{ es tangente a la superficie}} = 0$

$\Rightarrow \mathcal{P}^{(vel)} = 0$ : $\boxed{E = cte.}$

b) $E = \frac{1}{2}m\vec{v}^2 + U_g = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + mgz$

$L_z = m\rho^2\dot{\varphi}$ : $\dot{\varphi} = \frac{L_z}{m\rho^2}$

$z = \alpha\rho^2$ : $\dot{z} = 2\alpha\rho\dot{\rho}$

$\longrightarrow$ sustituyendo en E :

$$E = \frac{1}{2} m \dot{\rho}^2 \left(1 + 4\alpha^2\rho^2\right) + \frac{L_z^2}{2m\rho^2} + mg\alpha\rho^2$$

Condiciones Iniciales : $\rho = \rho_0$, $\vec{v} = v_0 \hat{e}_\varphi$ : $\begin{cases} \dot{\rho}(0) = 0 \\ \dot{\varphi}(0) = v_0/\rho_0 \\ \dot{z}(0) = 0 \end{cases}$

Luego : $L_z = L_z(0) = m\rho_0 v_0$

$$E = E(0) = \frac{1}{2} m v_0^2 + mg\alpha\rho_0^2$$

$$\Rightarrow \frac{1}{2} m v_0^2 + mg\alpha\rho_0^2 = \frac{1}{2} m \dot{\rho}^2 (1 + 4\alpha^2\rho^2) + \frac{1}{2} m v_0^2 \left(\frac{\rho_0}{\rho}\right)^2 + mg\alpha\rho^2$$

$$\boxed{\dot{\rho}^2 = \underbrace{2\,\frac{\left[\frac{1}{2} m v_0^2 \left(1 - (\rho_0/\rho)^2\right) + mg\alpha(\rho_0^2 - \rho^2)\right]}{m(1 + 4\alpha^2\rho^2)}}_{f(\rho)}}$$

c) $\dot{\rho} = 0$ para los valores extremos $\leftrightarrow f(\rho) = 0$ :

$$\frac{1}{2} m v_0^2 \left(1 - (\rho_0/\rho)^2\right) + mg\alpha(\rho_0^2 - \rho^2) = 0 \;!$$

$$(\rho_0^2 - \rho^2)\left(g\alpha - \frac{v_0^2}{2\rho^2}\right) = 0 \quad \begin{cases} \rho = \rho_0 \quad (\rho_{max}) \\ \rho = \frac{v_0}{\sqrt{2g\alpha}} = \frac{1}{\sqrt{2}}\rho_0 \quad (\rho_{min}) \end{cases}$$

$v_0 = \sqrt{g\alpha}\,\rho_0$

d) Si los valores extremos son iguales, $\rho = cte$ :

$$\rho_0 = \frac{v_0}{\sqrt{2g\alpha}} \; ; \quad \boxed{v_0 = \sqrt{2g\alpha}\,\rho_0}$$

(idem : $df/d\rho = 0 \leftrightarrow \ddot{\rho} = 0$)