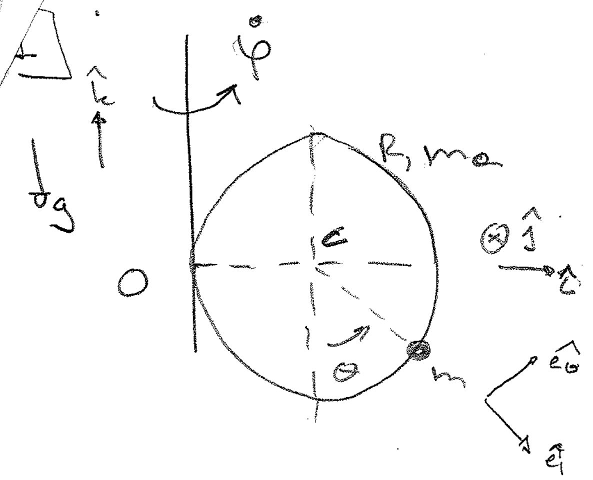
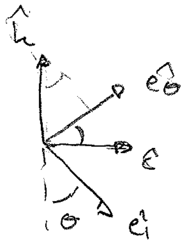



$\hat{k} = -\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_\theta$

$\hat{c} = \sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_\theta$

a) $\vec{L}_O^{\,T} = \vec{L}_O^{\,A} + \vec{L}_O^{\,m}$

masa $\quad \vec{r} = R\hat{c} + R\hat{e}_1$

$\vec{v}_p = R\dot{\hat{c}} + R\dot{\hat{e}}_1$

$\vec{v}_p = R\dot\varphi\,\hat{j} + R(\dot\varphi\sin\theta\,\hat{j} + \dot\theta\,\hat{e}_\theta)$

$\dot{\hat{c}} = \dot\varphi\,\hat{k}\wedge\hat{c} = \dot\varphi\,\hat{j}$

$\dot{\hat{j}} = \dot\varphi\,\hat{k}\wedge\hat{j} = -\dot\varphi\,\hat{c}$

$\dot{\hat{e}}_1 = (\dot\varphi\,\hat{k} - \dot\theta\,\hat{j})\wedge\hat{e}_1 = \dot\varphi\sin\theta\,\hat{j} - \dot\theta(-\hat{e}_\theta)$

$\dot{\hat{e}}_1 = \dot\varphi\sin\theta\,\hat{j} + \dot\theta\,\hat{e}_\theta$

$\dot{\hat{e}}_\theta = (\dot\varphi\,\hat{k} - \dot\theta\,\hat{j})\wedge\hat{e}_\theta = -\dot\varphi\cos\theta(-\hat{j}) - \dot\theta\,\hat{e}_1$

$\dot{\hat{e}}_\theta = \dot\varphi\cos\theta\,\hat{j} - \dot\theta\,\hat{e}_1$

$\vec{L}_O^{\,m} = \vec{r}\wedge m\vec{v}_p$

$= (R\hat{c} + R\hat{e}_1)\wedge m\left[R\dot\varphi(1+\sin\theta)\hat{j} + R\dot\theta\,\hat{e}_\theta\right]$

$= mR^2(\dot\varphi(1+\sin\theta)\hat{k} + mR^2\dot\theta\sin\theta(-\hat{j}) + mR^2\dot\varphi(1+\sin\theta)\hat{e}_\theta - mR^2\dot\theta\,\hat{j}$

$= mR^2\left[\dot\varphi(1+\sin\theta)\hat{k} - \dot\theta(1+\sin\theta)\hat{j} + \dot\varphi(1+\sin\theta)\hat{e}_\theta\right]$

---

ARO $\quad \mathbb{I}_C^{\{\hat{c},\hat{j},\hat{k}\}} = m_a R^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

por completitud se agrega esta parte pero $m_a \to 0$

$\mathbb{I}_O = \mathbb{I}_C + \mathbb{I}_p^{(M,C)} \qquad \vec{r}_C - \vec{r}_O = R\hat{c} \qquad \mathbb{I}_p^{M,G} = m_a R^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbb{I}_O = m_a R^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$

$\vec{L}_O^{\,A} = \mathbb{I}_O\vec{\omega}_A \qquad \vec{\omega}_A = \dot\varphi\,\hat{k} \Rightarrow \vec{L}_O^{\,A} = \frac{3}{2} m_a R^2\dot\varphi\,\hat{k}$

$$\Rightarrow \dot{\vec{L}}_0^T = R^2\left[\left(m(1+\sin\theta)\dot\varphi + \frac{3}{2}m_a\ddot\varphi\right)\hat{k} - m\ddot\theta(1+\sin\theta)\hat{j} + m\ddot\varphi(1+\sin\theta)\hat{e_\theta}\right]$$

b) $\vec{M}_O^{ext} = R\hat{e}\wedge(-m_a g\hat{k}) + (R\hat{e}+R\hat{e_\theta})\wedge(-mg\hat{k}) + M_1^{react}\hat{e} + M_2^{react}\hat{j}$

Ya que las Fuerzas entre la masa y el Aro forman un par de acción y reacción y se cancelan mutuamente.
De otra forma, son internas al sistema total.

$$\vec{M}_O^{ext} = m_a gR\hat{j} + mgR\hat{j} + mgR\sin\theta\,\hat{j} + \overbrace{M_1^{react}\hat{e} + M_2^{react}\hat{j}}^{\text{giro Libre según }\hat{k}}$$

$$\left.\begin{aligned} &\Rightarrow \vec{M}_O^{ext}\cdot\hat{k} = 0 \quad \Rightarrow \quad \frac{d\vec{L}_0}{dt}\cdot\hat{k} = \vec{M}_O^{ext}\cdot\hat{k} = 0 \\ &\frac{d(\vec{L}_0\cdot\hat{k})}{dt} = \frac{d\vec{L}_0}{dt}\cdot\hat{k} + \vec{L}_0\cdot\cancelto{0}{\frac{d\hat{k}}{dt}} = \frac{d\vec{L}_0}{dt}\cdot\hat{k} \quad \hat{k}\text{ es fijo}\end{aligned}\right] \Rightarrow \frac{d(\vec{L}_0\cdot\hat{k})}{dt} = 0$$

$$\Rightarrow \vec{L}_0\cdot\hat{k} = L_z \quad \text{cte.}$$

$$L_z = R^2\left(m(1+\sin\theta)\dot\varphi + \frac{3}{2}m_a\dot\varphi\right) + R^2 m\dot\varphi(1+\sin\theta)\overbrace{\hat{e_\theta}\cdot\hat{k}}^{\sin\theta}$$

$$L_z = R^2\left(m\dot\varphi(1+\sin\theta)^2 + \frac{3}{2}m_a\dot\varphi\right)$$

Ahora $m_a \to 0 \Rightarrow \boxed{L_z = mR^2\dot\varphi(1+\sin\theta)^2}$

c) Fuerzas conservativas: Peso

$$\frac{dE}{dt} = P^{res} \qquad P^{res} = P_{Normal}$$

tengo fuerzas Normales actuando sobre el Aro ($\vec{N}_A$) y sobre la partícula ($\vec{N}_P$) por acción y reacción $\vec{N}_A = -\vec{N}_P$

$\vec{v}_P = \vec{v}' + \vec{v}_T$ (Roverbal) y $\vec{v}_T = \vec{v}_A$

$$\Rightarrow P_{Normal} = \underbrace{\vec{N}_A\cdot\vec{v}_A}_{\text{aro}} + \vec{N}_P\cdot\vec{v}_P$$

$$\Rightarrow P_{Normal} = \vec{N}_A\cdot(\vec{v}_A - \vec{v}' - \vec{v}_A) = \vec{N}_A\cdot\vec{v}' = 0 \quad \Rightarrow P^{res} = 0 \quad E \text{ cte}$$

$$T = \frac{1}{2}m\vec{v}_P^{\,2} + \cancelto{0 \text{ pues } m_a\to 0}{\frac{1}{2}\vec{\omega}_A\cdot\mathbb{I}_O\vec{\omega}_A} \Rightarrow T = \frac{1}{2}m\vec{v}_P^{\,2}$$

$$U_g = -mgR\cos\theta$$

$$\vec{v}_P = R(1+\sin\theta)\hat{j} + R\dot\theta\hat{e_\theta}$$

$$T = \frac{1}{2}mR^2(1+\sin\theta)^2\dot\varphi^2 + \frac{1}{2}mR^2\dot\theta^2$$

$$E = E_0 = \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \dot{\varphi}^2 (1+\sin\theta)^2 \right) - mgR\cos\theta$$

$$E_0 = \frac{1}{2} m R^2 \dot{\varphi}_0^{\,2} - mgR$$

de la conservación de $L_z$:

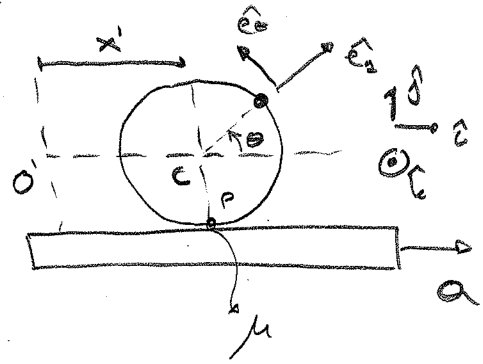
$$L_z = mR^2 \dot{\varphi} (1+\sin\theta)^2 \quad , \quad L_z = mR^2 \dot{\varphi}_0$$

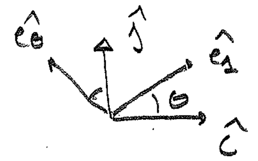
$$\rightarrow \dot{\varphi} = \frac{\dot{\varphi}_0}{(1+\sin\theta)^2}$$

c) $$E_0 = \frac{1}{2} m R^2 \left( \dot{\theta}^2 + \frac{\dot{\varphi}_0^{\,2}}{(1+\sin\theta)^4} (1+\sin\theta)^2 \right) - mgR\cos\theta = \frac{1}{2} m R^2 \dot{\varphi}_0^{\,2} - mgR$$

$$\dot{\theta}^2 = \dot{\varphi}_0^{\,2} \left( 1 - \frac{1}{(1+\sin\theta)^2} \right) + \frac{2g}{R} (\cos\theta - 1)$$

$\boxed{2}$




$$\hat{e}_1 = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

a) El centro de masa del rígido coincide con la partícula (el disco tiene masa despreciable)

$$\vec{r}_G = \vec{r}_C + R\hat{e}_1 \qquad \dot{\hat{e}}_1 = \vec{\omega}\wedge\hat{e}_1 = \dot{\theta}\hat{k}\wedge\hat{e}_1 = \dot{\theta}\,\hat{e}_\theta$$

$$\text{con} \quad \vec{r}_C = \vec{r}_{O'} + x'\hat{i} \qquad \dot{\hat{e}}_\theta = \vec{\omega}\wedge\hat{e}_\theta = \dot{\theta}\hat{k}\wedge\hat{e}_\theta = -\dot{\theta}\,\hat{e}_1$$

$$\vec{v}_G = \vec{v}_C + R\dot{\hat{e}}_1 = \vec{v}_{O'} + \dot{x}'\hat{i} + R\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a}_G = \vec{a}_{O'} + \ddot{x}'\hat{i} + R\ddot{\theta}\,\hat{e}_\theta + R\dot{\theta}\,\dot{\hat{e}}_\theta$$

O' es solidario a la placa $\Rightarrow \vec{a}_{O'} = a\hat{i}$

$$\vec{a}_G = a\hat{i} + \ddot{x}'\hat{i} + R\ddot{\theta}\,\hat{e}_\theta - R\dot{\theta}^2\,\hat{e}_1$$

RSD $\quad \vec{v}_P = 0 \quad$ (en el sistema solidario a la placa)

Dist. velocidades $\quad \vec{v}_P = \vec{v}_C + \vec{\omega}\wedge(\vec{r}_P - \vec{r}_C) \qquad \vec{r}_P - \vec{r}_C = -R\hat{j}$

$$\vec{v}_P = \dot{x}'\hat{i} + \dot{\theta}\hat{k}\wedge(-R)\hat{j} = (\dot{x}' + R\dot{\theta})\hat{i} = 0 \Rightarrow \dot{x}' = -R\dot{\theta}$$

$$\downarrow$$

$$\ddot{x}' = -R\ddot{\theta}$$

En el sistema absoluto, $\vec{v}_P = \vec{v}_{O'}$ punto de la placa

Dist velocidades

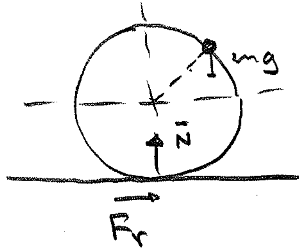
$$\vec{v}_P = \vec{v}_C + \dot{\theta}\hat{k}\wedge(-R\hat{j}) \quad , \quad \vec{v}_C = \vec{v}_{O'} + \dot{x}'\hat{i}$$

$$\Rightarrow \vec{v}_P = \vec{v}_{O'} + \dot{x}'\hat{i} + R\dot{\theta}\hat{i} = \vec{v}_{O'}$$

$\Rightarrow \dot{x}' = -R\dot{\theta}$ mismo resultado

$$\boxed{\vec{a}_G = a\hat{i} - R\ddot{\theta}\hat{i} + R\ddot{\theta}\,\hat{e}_\theta - R\dot{\theta}^2\,\hat{e}_1}$$

b)




1ra Cardinal $\quad m\vec{a}_G = N\hat{j} + F_r\hat{i} - mg\hat{j}$

$$\vec{a}_G = a\hat{\imath} - R\ddot{\theta}\hat{\imath} + R\ddot{\theta}(-\sin\theta\hat{\imath}+\cos\theta\hat{\jmath}) - R\dot{\theta}^2(\cos\theta\hat{\imath}+\sin\theta\hat{\jmath})$$

$1^{ra}$ cardinal $\hat{\imath})\ m(a - R\ddot{\theta}(1+\sin\theta) - R\dot{\theta}^2\cos\theta) = F_r \qquad (1)$

$$\hat{\jmath})\ mR(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = N - mg \qquad (2)$$

$2^{da}$ Cardinal en $C$ ($C$ tiene aceleración)

$$\vec{L}_C = m(\vec{r}_G - \vec{r}_C)\wedge\vec{v}_C + \mathbb{I}_C\vec{\omega}$$

$$\vec{r}_G - \vec{r}_C = R\hat{e}_1 = R\cos\theta\hat{\imath} + R\sin\theta\hat{\jmath}$$

$$\vec{v}_C = \vec{v}_{O'} + \dot{x}'\hat{\imath} \qquad \vec{a}_C = \vec{a}_{O'} + \ddot{x}'\hat{\imath} = a\hat{\imath} + \ddot{x}'\hat{\imath}$$

$$\frac{d\vec{L}_C}{dt} = m(\vec{v}_G - \vec{v}_C)\wedge\vec{v}_C + m(\vec{r}_G - \vec{r}_C)\wedge\vec{a}_C + \mathbb{I}_C\dot{\vec{\omega}}$$

La $2^{da}$ cardinal: $\frac{d\vec{L}_C}{dt} = m\vec{v}_G\wedge\vec{v}_C + \vec{M}_C^{(ext)}$

Juntando ambas: $m(\vec{r}_G - \vec{r}_C)\wedge\vec{a}_C + \mathbb{I}_C\dot{\vec{\omega}} = \vec{M}_C^{ext}$

$$m(\vec{r}_G - \vec{r}_C)\wedge\vec{a}_C = -mR\sin\theta(a+\ddot{x}')\hat{k}$$

$$\mathbb{I}_C^{\{\hat{e}_1,\hat{e}_\theta,\hat{k}\}} = mR^2\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad \vec{\omega} = \dot{\theta}\hat{k} \qquad \dot{\vec{\omega}} = \ddot{\theta}\hat{k}$$

$$\Rightarrow \vec{M}_C^{ext} = (mR^2\ddot{\theta} - mR(a+\ddot{x}')\sin\theta)\hat{k}$$

de la dinámica

$$\vec{M}_C^{ext} = -mgR\cos\theta\hat{k} + F_rR\hat{k}$$

$$\Rightarrow mR^2\ddot{\theta} - mR(a+\ddot{x}')\sin\theta = RF_r - mgR\cos\theta \qquad (3)$$

Sustituyendo $F_r$ de (1) en (3)

$$mR\left(R\ddot{\theta} - (a+\ddot{x}')\sin\theta\right) = mR\left(a - R\ddot{\theta}(1+\sin\theta) - R\dot{\theta}^2\cos\theta - g\cos\theta\right)$$

sustituyo $\ddot{x}' = -R\ddot{\theta}$ Ecuación de movimiento.

c) En eq. relativo $\dot{x}'=0$ $\dot{\theta}'=0$ y $\ddot{x}'=0$ $\ddot{\theta}=0$

en la ec. de movimiento.

$$-mRa\sin\theta = mR(a - g\cos\theta) \rightarrow -a\sin\theta - a = -g\cos\theta$$

$$\rightarrow a(1+\sin\theta) = g\cos\theta \quad \text{en } \theta = \frac{\pi}{4} \quad \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{a}{g} = \frac{\cos\theta}{1+\sin\theta} \rightarrow \boxed{\frac{a}{g} = \frac{1}{\sqrt{2}+1}}$$

Imponiendo el equilibrio en (1) y (2)

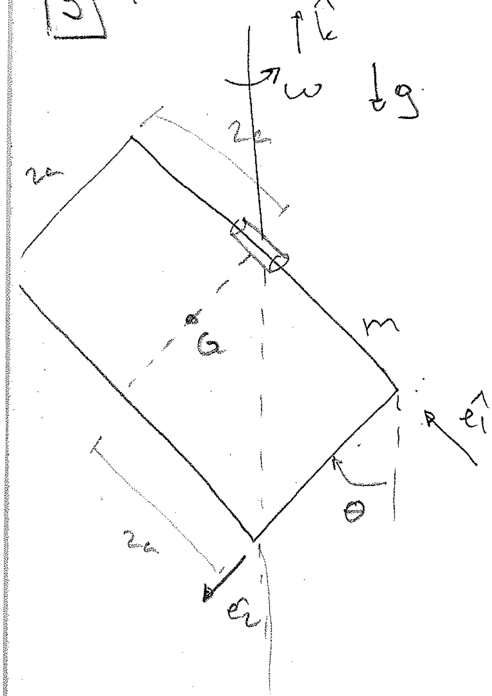
$F_r = ma$

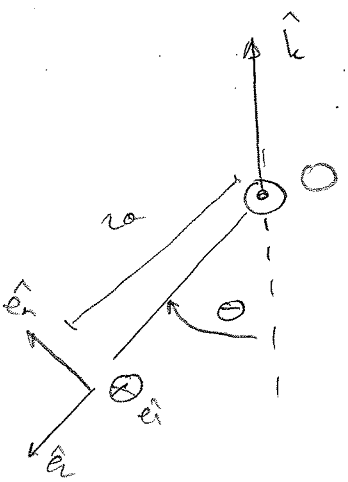
$N = mg$

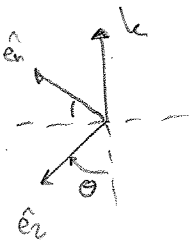
No desliza si $|\vec{F}_r| \leq \mu|\vec{N}|$

$$\rightarrow a \leq \mu g \rightarrow \boxed{\frac{a}{g} \leq \mu}$$

3




$$\hat{k} = -\cos\theta\,\hat{e}_2 + \sin\theta\,\hat{e}_n$$

$$\vec{\omega} = \omega\hat{k} + \ddot{\theta}\,\hat{e}_1$$

$$= -\omega\cos\theta\,\hat{e}_2 + \omega\sin\theta\,\hat{e}_n + \dot{\theta}\,\hat{e}_1$$

a) $\vec{L}_O = \mathbb{I}_O\vec{\omega}$ $\qquad \mathbb{I}_O = \mathbb{I}_G + \mathbb{J}_O^{m,G}$

$\vec{r}_G - \vec{r}_O = a\,\hat{e}_2$

$$\mathbb{I}_G^{\{\hat{e}_1,\hat{e}_2,\hat{e}_n\}} = \frac{m}{12}\begin{pmatrix} (2a)^2 & 0 & 0 \\ 0 & (4a)^2 & 0 \\ 0 & 0 & (4a)^2+(2a)^2 \end{pmatrix}$$

$$= \frac{ma^2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\mathbb{J}_O^{m,G\{\hat{e}_1,\hat{e}_2,\hat{e}_n\}} = ma^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \mathbb{I}_O^{\{\hat{e}_1,\hat{e}_2,\hat{e}_n\}} = ma^2\begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{pmatrix}$$

$$\vec{L}_O = \frac{4ma^2}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ -\omega\cos\theta \\ \omega\sin\theta \end{pmatrix}$$

$$\vec{L}_O = \frac{4}{3}ma^2\left(-\omega\cos\theta\,\hat{e}_2 + 2\omega\sin\theta\,\hat{e}_n\right) + \frac{4}{3}ma^2\,\dot{\theta}\,\hat{e}_1$$

b) $\dot{\vec{L}}_O = \vec{M}_O^{ext}$

$$\dot{\vec{L}}_O = \frac{4}{3} m a^2 \Big( + \omega \dot{\theta} \sin\theta \, \hat{e}_2 - \omega \cos\theta \, \dot{\hat{e}}_2 + 2\omega \dot{\theta} \cos\theta \, \hat{e}_n + 2\omega \sin\theta \, \dot{\hat{e}}_n$$
$$+ \ddot{\theta} \hat{e}_1 + \dot{\theta} \dot{\hat{e}}_1 \Big)$$

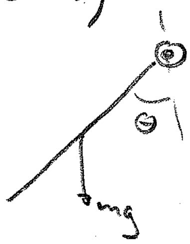
$$\dot{\hat{e}}_1 = \vec{\omega} \wedge \hat{e}_1 = (-\omega\cos\theta \, \hat{e}_2 + \omega\sin\theta \, \hat{e}_n + \dot{\theta} \hat{e}_1) \wedge \hat{e}_1 = + \omega\cos\theta \, \hat{e}_n + \omega \sin\theta \, \hat{e}_2$$

$$\dot{\hat{e}}_2 = (-\omega\cos\theta \, \hat{e}_2 + \omega\sin\theta \, \hat{e}_n + \dot{\theta} \hat{e}_1) \wedge \hat{e}_2 = -\omega\sin\theta \, \hat{e}_1 + \dot{\theta} \hat{e}_n$$

$$\dot{\hat{e}}_n = (-\omega\cos\theta \, \hat{e}_2 + \omega\sin\theta \, \hat{e}_n + \dot{\theta} \hat{e}_1) \wedge \hat{e}_n = -\omega\cos\theta \, \hat{e}_1 - \dot{\theta} \hat{e}_2$$

$$\dot{\vec{L}}_O = \frac{4}{3} m a^2 \Big( \omega \dot{\theta} \sin\theta \, \hat{e}_2 + \omega^2 \cos\theta \sin\theta \, \hat{e}_1 + 2\omega \dot{\theta} \cos\theta \, \hat{e}_n$$
$$+ 2\omega \sin\theta (-\omega\cos\theta) \, \hat{e}_1 + \ddot{\theta} \hat{e}_1 + \dot{\theta} (\omega\cos\theta \, \hat{e}_n + \omega\sin\theta \, \hat{e}_2)$$
$$- \omega \dot{\theta} \cos\theta \, \hat{e}_n - 2\omega \dot{\theta} \sin\theta \, \hat{e}_2 \Big)$$

$$\vec{M}_O^{ext} = -mga \sin\theta \, \hat{e}_1 + M_2 \hat{e}_2 + M_n \hat{e}_n$$




No hay momentos reactivos según $\hat{e}_1$

Ec de mov.: $\dot{\vec{L}}_O \cdot \hat{e}_1 = \vec{M}_O^{ext} \cdot \hat{e}_1$

$$\frac{4}{3} m a^2 \left( \ddot{\theta} - \omega^2 \sin\theta \cos\theta \right) = -mga \sin\theta$$

$$\ddot{\theta} - \omega^2 \sin\theta \cos\theta + \frac{3}{4} \frac{g}{a} \sin\theta = 0$$ ec. del tipo $\ddot{\theta} + f(\theta) = 0$

puntos de eq $\theta_{eq}$ t.q $f(\theta_{eq}) = 0$ si $\left.\frac{df}{d\theta}\right|_{\theta_{eq}} > 0$ estables

$\left.\frac{df}{d\theta}\right|_{\theta_{eq}} < 0$ inestables

$$f(\theta) = \sin\theta \left( \frac{3}{4} \frac{g}{a} - \omega^2 \cos\theta \right) \rightarrow \theta_1 = 0$$
$\theta_2 = \pi$

$\theta_3$ t.q $\cos\theta_3 = \frac{3}{4} \frac{g}{a} \frac{1}{\omega^2}$

$\theta_3$ $\exists$ sii $\frac{3}{4} \frac{g}{a} \frac{1}{\omega^2} < 1$

$$\frac{df}{d\theta} = \cos\theta\left(\frac{3}{4}\frac{g}{a} - \omega^2\cos\theta\right) + \omega^2\sin^2\theta$$

$$\left.\frac{df}{d\theta}\right|_{\theta_1=0} = \frac{3}{4}\frac{g}{a} - \omega^2 \overset{?}{>} 0 \text{ si } \frac{3}{4}\frac{g}{a}\frac{1}{\omega^2} > 1 \quad \text{estable}$$

$$\text{si } \frac{3}{4}\frac{g}{a}\frac{1}{\omega^2} < 1 \quad \text{inestable.}$$

$$\left.\frac{df}{d\theta}\right|_{\theta_2=\pi} = -1\left(\frac{3}{4}\frac{g}{a} + \omega^2\right) < 0 \quad \text{siempre} \quad \text{inestable siempre}$$

$$\left.\frac{df}{d\theta}\right|_{\theta_3} = \omega^2\sin^2\theta_3 > 0 \quad \text{Estable}$$

$\Rightarrow \theta_3$ cuando Existe es estable.