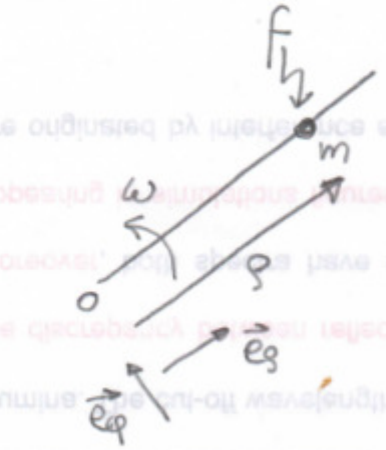
Ejercicio N°1

Primer Parcial Mecánica Newtoniana (1122)
11/5/2021



$\rho(0) = L \quad \dot{\rho}(0) = -v_0$

parte a: $m\vec{a} = T\vec{e}_\rho + N\vec{e}_\varphi$

$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\vec{e}_\varphi$

$\dot{\varphi} = \omega \quad \ddot{\varphi} = 0$

$m\ddot{\rho} - m\rho\omega^2 = T > 0$ porque $\dot{\rho} < 0$

$N = 2m\dot{\rho}\omega < 0$

$|T| = f|N| \Rightarrow T = -2fm\dot{\rho}\omega$

$\Rightarrow m\ddot{\rho} + 2fm\dot{\rho}\omega - m\rho\omega^2 = 0 \Rightarrow \boxed{\ddot{\rho} + 2f\omega\dot{\rho} - \rho\omega^2 = 0}$

parte b: $\rho = C e^{\lambda t} \Rightarrow \lambda^2 + 2f\omega\lambda - \omega^2 = 0$

$\Rightarrow \lambda_\pm = \dfrac{-2f\omega \pm \sqrt{4f^2\omega^2 + 4\omega^2}}{2} \Rightarrow \lambda_\pm = \omega\left(-f \pm \sqrt{f^2+1}\right)$

$\lambda_+ > 0, \lambda_- < 0$

$\rho(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}$

$\rho(0) = L = C_+ + C_- \Rightarrow C_+ = L - C_-$

$\dot{\rho}(t) = C_+\lambda_+ e^{\lambda_+ t} + C_-\lambda_- e^{\lambda_- t} \Rightarrow \dot{\rho}(0) = -v_0 = C_+\lambda_+ + C_-\lambda_-$

$-v_0 = L\lambda_+ + C_-(\lambda_- - \lambda_+) \Rightarrow C_- = \dfrac{v_0 + L\lambda_+}{\lambda_+ - \lambda_-} \qquad \lambda_+ - \lambda_- = 2\omega\sqrt{f^2+1}$

$C_+ = L - \dfrac{v_0 + L\lambda_+}{\lambda_+ - \lambda_-} = -\dfrac{v_0 + L\lambda_-}{\lambda_+ - \lambda_-}$

$\boxed{\rho(t) = -\dfrac{v_0 + L\lambda_-}{\lambda_+ - \lambda_-} e^{\lambda_+ t} + \dfrac{v_0 + L\lambda_+}{\lambda_+ - \lambda_-} e^{\lambda_- t}}$

parte c: $\rho(t) = 0 \Rightarrow (v_0 + L\lambda_-)e^{\lambda_+ t} = (v_0 + L\lambda_+)e^{\lambda_- t}$

$e^{(\lambda_+ - \lambda_-)t} = \dfrac{v_0 + L\lambda_+}{v_0 + L\lambda_-} \Rightarrow \boxed{t = \dfrac{1}{\lambda_+ - \lambda_-} \ln \dfrac{v_0 + L\lambda_+}{v_0 + L\lambda_-}}$

parte d: Para que t exista $\dfrac{v_0 + L\lambda_+}{v_0 + L\lambda_-} > 0$

$v_0, \lambda_+ > 0 \Rightarrow v_0 + L\lambda_+ > 0 \Rightarrow v_0 + L\lambda_- > 0 \Rightarrow \boxed{v_0 > L\omega\left(f + \sqrt{f^2+1}\right)}$

parte e:

Si $v_0 = -L\lambda_-$ $t \to \infty$ y $C_+ = 0$ $\quad \dot{\rho}(t) = C_-\lambda_- e^{\lambda_- t} \to 0 \ (\lambda_- < 0)$

$C_- = L$

$P = T\vec{e}_\rho \cdot \vec{v} = T\vec{e}_\rho \cdot (\dot{\rho}\vec{e}_\rho + \rho\omega\vec{e}_\varphi) = T\dot{\rho} = -2fm\dot{\rho}^2\omega$

$$W_T = \int_0^t P(t)\,dt = -2fm\omega \int_0^\infty L^2\lambda_-^2 e^{2\lambda_- t} =$$

$$= -fm\omega L^2 \lambda_- e^{2\lambda_- t}\Big|_0^\infty = \boxed{-fm\omega^2 L^2\left(f+\sqrt{f^2+1}\right) = W_T < 0}$$

Verificación: $T(t)-T(0)=W_N+W_T$

$$T=\frac{1}{2} m\dot{\rho}^2+\frac{1}{2} m\rho^2\omega^2 \qquad T(t)=0$$

$$T(0)=\frac{1}{2} m v_0^2+\frac{1}{2} mL^2\omega^2$$

$$W_N=\int_0^t dt\,P_N \qquad P_N = N\vec{e}_\varphi\cdot\vec{v} = N\rho\omega = 2m\omega^2\rho\dot{\rho}$$

$$W_N=\int_0^t dt\,2m\omega^2\rho\dot{\rho} = m\omega^2\rho^2\Big|_L^0 = -m\omega^2L^2$$

$$W_T - m\omega^2L^2 = -\frac{1}{2} mv_0^2 - \frac{1}{2} mL^2\omega^2$$

$$W_T = -\frac{1}{2} mv_0^2 + \frac{1}{2} m\omega^2L^2 = -\frac{1}{2} mL^2\omega^2\left(f^2+f^2+\cancel{1}+2f\sqrt{f^2+1}\right) \cancel{+\frac{1}{2} m\omega^2L^2}$$

$$\boxed{W_T = -mL^2\omega^2 f\left(f+\sqrt{f^2+1}\right)} < 0$$

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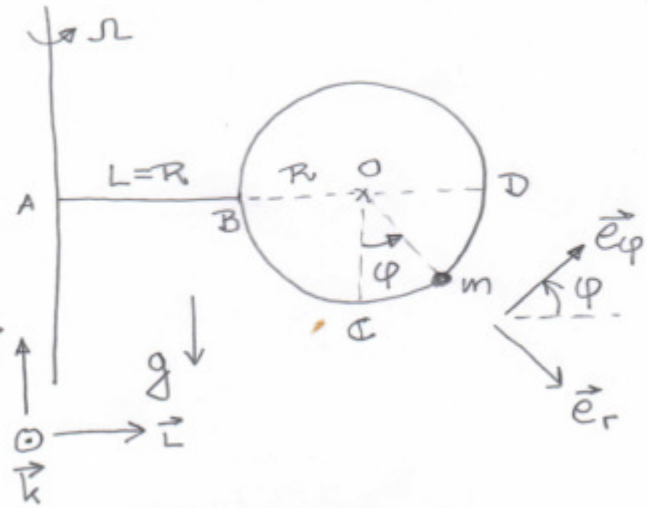
Ejercicio Nº2:



parte a:

$$m\vec{a} = -mg\vec{j} + N\vec{e}_r + B\vec{k}$$

$$\vec{a} = \vec{a}\,' + \vec{a}_T + \vec{a}_C$$

$$\vec{a}\,' = R\ddot{\varphi}\vec{e}_\varphi - R\dot{\varphi}^2\vec{e}_r$$

$$\vec{a}_T = \underbrace{\vec{a}_A}_{0} + \underbrace{\dot{\vec{\omega}}}_{0}\wedge(P-A) + \vec{\omega}\wedge[\vec{\omega}\wedge(P-A)]$$

$$P = A + 2R\vec{i} + R\vec{e}_r$$

$$\vec{\omega} = \Omega\vec{j}$$

$$\vec{\omega}\wedge(P-A) = 2R\Omega\underbrace{\vec{j}\wedge\vec{i}}_{-\vec{k}} + R\Omega\underbrace{\vec{j}\wedge\vec{e}_r}_{-\operatorname{sen}\varphi\vec{k}}$$

$$\vec{e}_r = \operatorname{sen}\varphi\,\vec{i} - \cos\varphi\,\vec{j}$$

$$\vec{a}_T = -R\Omega^2(2+\operatorname{sen}\varphi)\,\vec{j}\wedge\vec{k} = -R\Omega^2(2+\operatorname{sen}\varphi)\,\vec{i}$$

$$\vec{a}_C = 2\vec{\omega}\wedge\vec{v}\,' = 2\Omega\vec{j}\wedge R\dot{\varphi}\vec{e}_\varphi = -2\Omega R\dot{\varphi}\cos\varphi\,\vec{k}$$

$$\vec{e}_\varphi = \cos\varphi\,\vec{i} + \operatorname{sen}\varphi\,\vec{j}$$

$$\cancel{m}\vec{a}\cdot\vec{e}_\varphi = -\cancel{m}g\vec{j}\cdot\vec{e}_\varphi = -\cancel{m}g\operatorname{sen}\varphi$$

$$\boxed{R\ddot{\varphi} - R\Omega^2(2+\operatorname{sen}\varphi)\cos\varphi = -g\operatorname{sen}\varphi}$$

Verificación: Aplico teorema de la energía en el sistema relativo

1) $-mg\vec{j}$ conservativo: $U_g = mgy = -mgR\cos\varphi$

2) $N\vec{e}_r + B\vec{k} \perp \vec{v}\,' = R\dot{\varphi}\vec{e}_\varphi \Rightarrow$ potencia nula

3) $-m\vec{a}_C \perp \vec{v}\,' \Rightarrow$ también es de potencia nula

4) $-m\vec{a}_T = m\Omega^2\underbrace{R(2+\operatorname{sen}\varphi)}_{x}\vec{i} = m\Omega^2 x\vec{i}$

$$U_T = -\frac{m\Omega^2x^2}{2} \Rightarrow \vec{F}_T = -\frac{\partial U_T}{\partial x}\vec{i}$$

$$T' + U' = E \Rightarrow \frac{mR^2\dot{\varphi}^2}{2} - mgR\cos\varphi - \frac{m\Omega^2}{2}R^2(2+\operatorname{sen}\varphi)^2 = E$$

$$\cancel{m}R^{\cancel{2}}\cancel{\dot{\varphi}}\ddot{\varphi} + \cancel{m}gR\operatorname{sen}\varphi\cancel{\dot{\varphi}} - \cancel{m}\Omega^2R^{\cancel{2}}(2+\operatorname{sen}\varphi)\cos\varphi\cancel{\dot{\varphi}} = 0 \checkmark$$

parte b: $\dfrac{\partial U'}{\partial\varphi} = mgR\operatorname{sen}\varphi - m\Omega^2R^2(2+\operatorname{sen}\varphi)\cos\varphi$

Punto medio $\varphi = \frac{\pi}{4} \Rightarrow \operatorname{sen}\varphi = \cos\varphi = \frac{\sqrt{2}}{2}$

$$\left.\frac{\partial U'}{\partial\varphi}\right|_{\varphi=\frac{\pi}{4}} = \cancel{m}g\cancel{R}\frac{\cancel{\sqrt{2}}}{\cancel{2}} - \cancel{m}\Omega^2R^{\cancel{2}}\left(2+\frac{\sqrt{2}}{2}\right)\frac{\cancel{\sqrt{2}}}{\cancel{2}} = 0$$

$$\boxed{g = \frac{4+\sqrt{2}}{2}R\Omega^2}$$

$$\frac{\partial^2 U'}{\partial\varphi^2} = mgR\cos\varphi + m\Omega^2R^2(2+\operatorname{sen}\varphi)\operatorname{sen}\varphi - m\Omega^2R^2\cos^2\varphi$$

$$\left.\frac{\partial^2 U'}{\partial\varphi^2}\right|_{\varphi=\frac{\pi}{4}} = mgR\frac{\sqrt{2}}{2} + m\Omega^2R^2\left(\sqrt{2}+\cancel{\frac{1}{2}}\right) - \cancel{\frac{m\Omega^2R^2}{2}} > 0$$

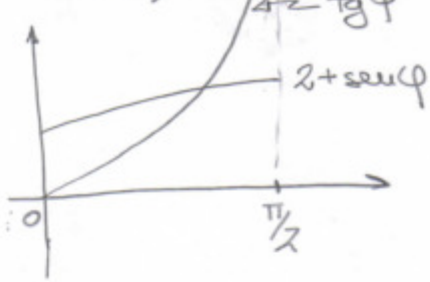
La posición es de equilibrio estable

parte c: $\varphi(0)=0,\ \dot\varphi(0)=0 \Rightarrow E = -mgR - m\Omega^2R^2\,2$

$$\frac{\cancel{m}R^{\cancel{2}}\dot\varphi^2}{2} = \cancel{m}gR(\cos\varphi-1) + \frac{\cancel{m}\Omega^2R^{\cancel{2}}}{2}\operatorname{sen}^2\varphi + 2\cancel{m}\Omega^2R^{\cancel{2}}\operatorname{sen}\varphi$$

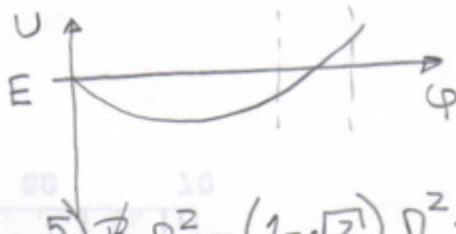
$\ddot\varphi(0) = 2\Omega^2 > 0 \Rightarrow$ la partícula efectivamente se mueve hacia D

La condición $\frac{\partial U'}{\partial\varphi}=0 \sim g\operatorname{tg}\varphi = \Omega R(2+\operatorname{sen}\varphi)$



$\frac{\partial U'}{\partial\varphi}=0$ en un único punto en $0\leq\varphi\leq\frac{\pi}{2}$

$\Rightarrow$ solo habrá un extremo relativo de U' en $0\leq\varphi\leq\frac{\pi}{2}$, que por la parte b es un mínimo $\Rightarrow$ alcanza $E\geq U$ en $\frac{\pi}{2}$ para que la partícula llegue.



$\sim \dot\varphi^2(\pi/2)\geq 0$

$$\dot\varphi^2(\pi/2) = \frac{2}{R}\left(-g+\Omega^2R\frac{5}{2}\right) = \frac{2}{\cancel{R}}\left(-\frac{4}{\cancel{2}}-\frac{\sqrt{2}}{\cancel{2}}+\frac{5}{2}\right)\cancel{R}\Omega^2 = (1-\sqrt{2})\Omega^2 < 0$$

La partícula no alcanza el punto D.

parte d: $P = (N\vec{e}_r + B\vec{k})\cdot\vec{v}$

$$\vec{v} = \vec{v}\,' + \vec{v}_T = R\dot\varphi\,\vec{e}_\varphi + \underbrace{\vec{v}_A}_{0} + \vec{\omega}\wedge(P-A) = R\dot\varphi\,\vec{e}_\varphi - R\Omega(2+\operatorname{sen}\varphi)\vec{k}$$

$$\left.\begin{array}{l} P = -BR\Omega(2+\operatorname{sen}\varphi) \\ B = m\vec{a}\cdot\vec{k} = -2m\Omega R\dot\varphi\cos\varphi \end{array}\right\} P = 2m\Omega^2R^2(2+\operatorname{sen}\varphi)\cos\varphi\,\dot\varphi$$

$$W_{\vec{R}} = \int_0^{\pi/4} P(t)\,dt = m\Omega^2R^2(2+\operatorname{sen}\varphi)^2\Big|_0^{\pi/4} = m\Omega^2R^2\left[\left(2+\frac{\sqrt{2}}{2}\right)^2-4\right]$$

$$W_{\vec{R}} = m\Omega^2R^2\left(2\sqrt{2}+\frac{1}{2}\right) = \boxed{\frac{m\Omega^2R^2}{2}(4\sqrt{2}+1) = W_{\vec{R}}}$$

Verificación: Aplico el teorema de la energía

$$T(\pi/4) + U_g(\pi/4) - T(0) - U_g(0) = W_{\vec{R}}$$

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$$T = \frac{mR^2\dot{\varphi}^2}{2} + \frac{mR^2\Omega^2}{2}\left(2+\operatorname{sen}\varphi\right)^2$$

$$T(\pi/4) = mg\left(\frac{\sqrt{2}}{2}-1\right) + \frac{m\Omega^2R^2}{2}\,\frac{1}{2} + m\Omega^2R^2\sqrt{2} + \frac{mR^2\Omega^2}{2}\left(2+\frac{\sqrt{2}}{2}\right)^2$$

$$U_g(\pi/4) = -mgR\frac{\sqrt{2}}{2}$$

$$T(0) = 2mR^2\Omega^2$$

$$U_g(0) = -mgR$$

$$W_{\vec{R}} = \frac{m\Omega^2R^2}{4}\left(1 + 4\sqrt{2} + 2\left(\cancel{4} + 2\sqrt{2} + \frac{1}{2}\right) - \cancel{8}\right) = \frac{m\Omega^2R^2}{4}\left(2+8\sqrt{2}\right)$$

$$\boxed{W_{\vec{R}} = \frac{m\Omega^2R^2}{2}\left(1+4\sqrt{2}\right)}\ \checkmark$$