Ejercicio 7

a) Se conservan (i) $\ell = \vec{L_A} \cdot \hat{k}$ (ya que $\vec{M}_A^{(ext)} \cdot \hat{k} = 0$) siendo $\vec{L_A}$ el momento angular del sistema visto desde A

(ii) E : energía del sistema (no trabajan agentes no conservativos)

$$\ell = I_A \dot{\varphi} + m r^2 \dot{\varphi} \; ; \; I_A = \int_0^{2L} \left(\frac{M}{2L}\right) x^2 dx = \frac{4}{3} M L^2 = m L^2$$

$$\Rightarrow \ell = m(r^2 + L^2)\dot{\varphi} \quad (i)$$

$$E = \frac{1}{2} I_A \dot{\varphi}^2 + \frac{1}{2} m (\dot{r}^2 + (r\dot{\varphi})^2) + \frac{1}{2} K r^2$$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r^2 + L^2)\dot{\varphi}^2 + \frac{1}{2} K r^2 \quad (ii)$$

de (i): $\dot{\varphi} = \dfrac{\ell}{m(r^2+L^2)}$; sustituyendo en (ii):

$$E = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2m(r^2+L^2)} + \frac{1}{2} K r^2 : \boxed{\dot{r}^2 = \underbrace{\frac{2}{m}\left[E - \frac{\ell^2}{2m(r^2+L^2)} - \frac{1}{2} K r^2\right]}_{f(r)}}$$

b) $\dot{\varphi}(0) = \omega_0$; $r(0) = L$; $\dot{r}(0) = 0$: $\begin{cases} \ell = 2 m L^2 \omega_0 \\ E = m L^2 \omega_0^2 + \frac{1}{2} K L^2 \end{cases}$ sustituyendo en $f(r)$:

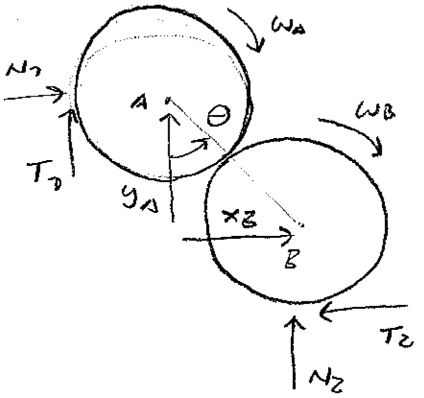
$$\dot{r}^2 = 2\omega_0^2 L^2 \left(1 - \frac{2L^2}{r^2+L^2}\right) + \frac{K}{m}(r^2 - L^2)$$

$$= 2\omega_0^2 L^2 \left(\frac{r^2 - L^2}{r^2+L^2}\right) + \frac{K}{m}(r^2 - L^2) = (r^2 - L^2)\frac{K}{m}\left[\frac{2\omega_0^2}{(K/m)} \frac{L^2}{r^2+L^2} - 1\right]$$

Para que la partícula escape : $\dot{r}^2 > 0$ si $r \in (L, 2L]$:

$$\dot{r}^2(r=2L) > 0 : \frac{2\omega_0^2}{K/m} \frac{\cancel{L^2}}{5\cancel{L^2}} - 1 > 0 : \boxed{\omega_0^2 > \frac{5}{2} K/m}$$

Ejercicio 2



a) rodadura sin deslizamiento: $\dot{y}_A = -\omega_A R$

$$\dot{x}_B = \omega_B R$$

$y_A = 2R\cos\theta$: $\dot{y}_A = -2R\,\text{sen}\,\theta\,\dot{\theta}$: $\omega_A = 2\,\text{sen}\,\theta\,\dot{\theta}$

$x_B = 2R\,\text{sen}\,\theta$: $\dot{x}_B = 2R\cos\theta\,\dot{\theta}$: $\omega_B = 2\cos\theta\,\dot{\theta}$

$$T = \frac{1}{2}m\dot{y}_A^2 + \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}m v_B^2 + \frac{1}{2}I_B\omega_B^2 = \frac{1}{2}m\,4R^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)4R^2\dot{\theta}^2$$

$\Rightarrow T = 3mR^2\dot{\theta}^2$

$U = mg\,2R\cos\theta$

$T + U = E\,(\text{cte.})$: $\boxed{3mR^2\dot{\theta}^2 + 2mgR\cos\theta = E}$ $\left(= 2mgR\cos\frac{\pi}{6} = mgR\sqrt{3}\right)$

b) 2ª C a disco de centro A en A: $RT_1 = I\dot{\omega}_A = \frac{mR^2}{2}\left(2\cos\theta\,\dot{\theta}^2 + 2\,\text{sen}\,\theta\,\ddot{\theta}\right)$ (i)

2ª C a disco de centro B en B: $RT_2 = I\dot{\omega}_B = \frac{mR^2}{2}\left(-2\,\text{sen}\,\theta\,\dot{\theta}^2 + 2\cos\theta\,\ddot{\theta}\right)$ (ii)

1era C al sistema: $N_1 - T_2 = 2m\ddot{x}_G = 2m\frac{\ddot{x}_B}{2} = m\left(-2R\,\text{sen}\,\theta\,\dot{\theta}^2 + 2R\cos\theta\,\ddot{\theta}\right)$ (iii)

$$T_1 + N_2 - 2mg = 2m\ddot{y}_G = 2m\frac{\ddot{y}_A}{2} = m\left(-2R\cos\theta\,\dot{\theta}^2 - 2R\,\text{sen}\,\theta\,\ddot{\theta}\right) \quad \text{(iv)}$$

Evaluando (i - iv) en t=0:

$$\begin{cases} T_1(0) = mR\,\text{sen}\frac{\pi}{6}\,\ddot{\theta}(0) & \text{(i')} \\ T_2(0) = mR\cos\frac{\pi}{6}\,\ddot{\theta}(0) & \text{(ii')} \\ N_1(0) - T_2(0) = 2mR\cos\frac{\pi}{6}\,\ddot{\theta}(0) & \text{(iii')} \\ T_1(0) + N_2(0) - 2mg = -2mR\,\text{sen}\frac{\pi}{6}\,\ddot{\theta}(0) & \text{(iv')} \end{cases}$$

A partir de la cons. de E (derivando): $\ddot{\theta} = \frac{1}{3}\,g/R\,\text{sen}\,\theta$: $\ddot{\theta}(0) = \frac{1}{3}\,g/R\,\text{sen}\frac{\pi}{6}$

$\Rightarrow T_1(0) = \frac{1}{3}mg\,\text{sen}^2\frac{\pi}{6} = \frac{mg}{12} \xrightarrow{(iv')} N_2(0) = mg\left(2 - \text{sen}^2\frac{\pi}{6}\right) = \frac{7}{4}mg$

$T_2(0) = \frac{1}{3}mg\,\text{sen}\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{12}mg \xrightarrow{(iii')} N_1(0) = mg\,\text{sen}\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}mg$

No deslizamiento:

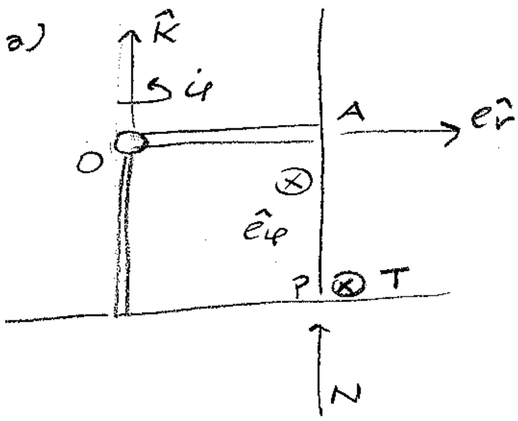
$T_1(0) \le f N_1(0)$: $\frac{mg}{12} \le f\frac{\sqrt{3}}{4}mg$: $f \ge \frac{1}{3\sqrt{3}}$

$T_2(0) \le f N_2(0)$: $\frac{\sqrt{3}}{12}mg \le f\frac{7}{4}mg$: $f \ge \frac{1}{7\sqrt{3}}$

$$f_{mín} = \frac{1}{3\sqrt{3}}$$

Ejercicio 3

a)



$$\vec{\omega} = \dot{\varphi}\hat{k} - \dot{\psi}\hat{e_r} \quad (\dot{\varphi}(0)=0,\ \dot{\psi}(0)=\dot{\psi}_0)$$

$$I_O^{\{\hat{e_r},\hat{e_\varphi},\hat{k}\}} = \begin{pmatrix} \frac{ma^2}{2} & 0 & 0 \\ 0 & \frac{ma^2}{4}+ma^2 & 0 \\ 0 & 0 & \frac{ma^2}{4}+ma^2 \end{pmatrix} = \frac{ma^2}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\vec{L_O} = \mathbb{I}_O \vec{\omega} = \frac{ma^2}{4}\left(5\dot{\varphi}\hat{k} - 2\dot{\psi}\hat{e_r}\right)$$

$$\dot{\vec{L_O}} = \vec{M_O}^{(ext)} = \underbrace{(P-O)}_{a\hat{e_r} - a\hat{k}} \times (fN\hat{e_\varphi} + N\hat{k}) + amg\,\hat{e_\varphi}$$

$$= afN\,\hat{e_r} + a(mg-N)\,\hat{e_\varphi} + afN\,\hat{k}$$

derivando $\vec{L_O}$: $\dot{\vec{L_O}} = \frac{ma^2}{4}\left(5\ddot{\varphi}\hat{k} - 2\ddot{\psi}\hat{e_r} - 2\dot{\psi}\dot{\varphi}\hat{e_\varphi}\right)$

$$\longrightarrow \begin{cases} afN = -\frac{ma^2}{2}\ddot{\psi} & (i) \\ a(mg-N) = -\frac{ma^2}{2}\dot{\psi}\dot{\varphi} & (ii) \\ afN = \frac{5}{4}ma^2\ddot{\varphi} & (iii) \end{cases}$$

Igualando en (i) y (iii): $\boxed{-\ddot{\psi} = \frac{5}{2}\ddot{\varphi}}$ (I)

de (i): $N = -\frac{ma\ddot{\psi}}{2f}$; sust. en (ii): $\boxed{a\left(mg + \frac{ma\ddot{\psi}}{2f}\right) = -\frac{ma^2}{2}\dot{\psi}\dot{\varphi}}$ (II)

b) Integrando (I) en el tiempo: $\underbrace{\dot{\psi} - \dot{\psi}(0)}_{\dot{\psi}_0} = -\frac{5}{2}(\dot{\varphi} - \underbrace{\dot{\varphi}(0)}_{0})$

$$\Rightarrow \boxed{\dot{\psi} = -\frac{5}{2}\dot{\varphi} + \dot{\psi}_0} \quad : \quad C_1 = -5/2, \quad C_2 = \dot{\psi}_0$$

$$\vec{v_P} = a(\dot{\varphi} - \dot{\psi})\hat{e_\varphi}$$

$$= a\left(\dot{\varphi} - \left(-\frac{5}{2}\dot{\varphi} + \dot{\psi}_0\right)\right)\hat{e_\varphi} = a\left(\frac{7}{2}\dot{\varphi} - \dot{\psi}_0\right)\hat{e_\varphi}$$

Comienza a rodar sin deslizar para: $v_P = 0$: $\boxed{\dot{\varphi} = \frac{2}{7}\dot{\psi}_0}$