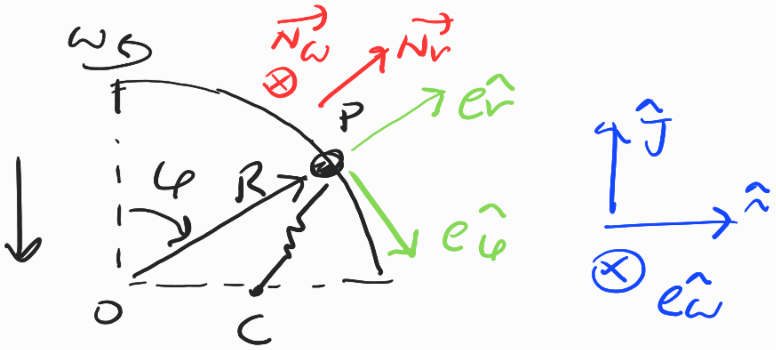
Mecánica Newtoniana, examen 16/02/24

Ej 1)



La reacción normal de la guía es $\vec{N} = N_r \hat{e_r} + N_\omega \hat{e_\omega}$

⇒ Obtendremos la ecuación de movimiento a partir de la proyección según $\hat{e_\varphi}$ de la 2da ley de Newton:

$$(\vec F = m\vec a) \cdot \hat{e_\varphi}$$

A efectos de hallar $\vec a \cdot \hat{e_\varphi}$ consideremos:

$S' : \{O, \hat i, \hat j, \hat{e_\omega}\}$ sistema solidario a la guía (gira con velocidad angular $\vec\omega = \omega \hat j$)

Teo. Coriolis: $\vec a = \vec a' + \vec{a_T} + \vec{a_C}$

$$\vec a' \cdot \hat{e_\varphi} = R\ddot\varphi$$

$$\vec{a_T} \cdot \hat{e_\varphi} = -\omega^2 R \operatorname{sen}\varphi \, \hat i \cdot \hat{e_\varphi} = -\omega^2 R \operatorname{sen}\varphi \cos\varphi$$

$$\vec{a_C} \cdot \hat{e_\varphi} = (2\vec\omega \times \vec v') \cdot \hat{e_\varphi} = 0 \quad (\vec v' = R\dot\varphi \hat{e_\varphi})$$

$$\Rightarrow m(R\ddot\varphi - \omega^2 R \operatorname{sen}\varphi \cos\varphi) = \vec F \cdot \hat{e_\varphi}$$

$$\vec{F_{el}} = -K(P-C) = -K(P-O+O-C) = -K\left(R\hat{e_r} - \frac{R}{2}\hat i\right)$$

$$\vec{F_{el}} \cdot \hat{e_\varphi} = K\frac{R}{2}\cos\varphi$$

$$\vec P = -mg\hat j \,; \quad \vec P \cdot \hat{e_\varphi} = mg \operatorname{sen}\varphi$$

$$\Rightarrow \boxed{\ddot{\varphi} - \omega^2 \sin\varphi\cos\varphi - \frac{1}{2}\frac{K}{m}\cos\varphi - \frac{g}{R}\sin\varphi = 0}$$

b) Preintegro la ecuación de movimiento ($\varphi(0)=0$, $\dot{\varphi}(0)=0$)

$$\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\omega^2\sin^2\varphi - \frac{1}{2}\frac{K}{m}\sin\varphi - g/R\,(1-\cos\varphi) = 0 \; ;$$

$\dot{\varphi}^2(\pi/2) = \omega^2 + K/m + 2g/R$; luego, a partir del Teo. de Roberval ($\vec{v} = \vec{v}\,' + \vec{v}_T$):

$$\boxed{\vec{v}(\pi/2) = R\,\dot{\varphi}(\pi/2)\,\hat{e}_\varphi + \omega R\,\hat{e}_\omega}$$

c) $\mathcal{P}_N = \vec{N}\cdot\vec{v} = N_\omega v_T$, $\vec{v}_T = \omega R\sin\varphi\,\hat{e}_\omega$

$N_\omega = m\vec{a}\,'\cdot\hat{e}_\omega = m a_c$, $\vec{a}_c = 2\vec{\omega}\times\vec{v}\,'$

$N_\omega = m 2\omega R\cos\varphi\,\dot{\varphi}$

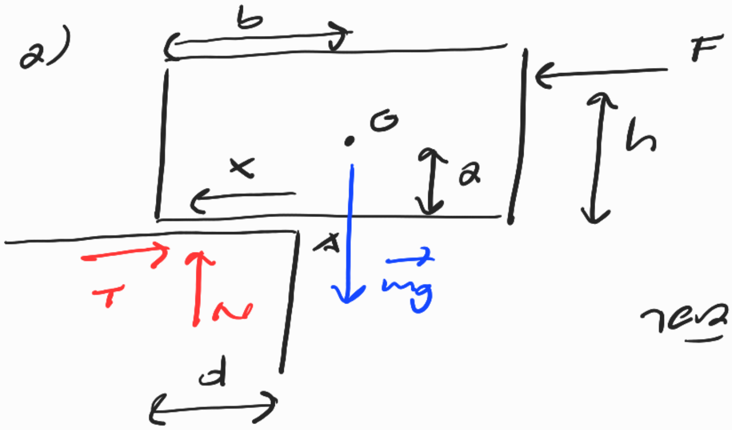
Luego:

$$W_N = \int_0^{t_{\pi/2}} \mathcal{P}_N\, dt = \int_0^{t_{\pi/2}} 2m\omega^2R^2\sin\varphi\cos\varphi\,\dot{\varphi}\, dt =$$

$$\int_0^{\pi/2} d\varphi\; 2m\omega^2R^2\sin\varphi\cos\varphi = m\omega^2R^2\sin^2\varphi\Big|_0^{\pi/2} \; :$$

$$\boxed{W_N = m\omega^2R^2}$$

Ej 2) a)



1era Cardinal : $\begin{cases} T = F \\ N = mg \end{cases}$

2da Cardinal desde A : $xN + mg(b-d) = hF \xrightarrow{N=mg}$

$$x = h\,F/mg - (b-d)$$

Condiciones de permanencia en el equilibrio:

i) no desprendimiento de la placa: $N = mg > 0$ ✓

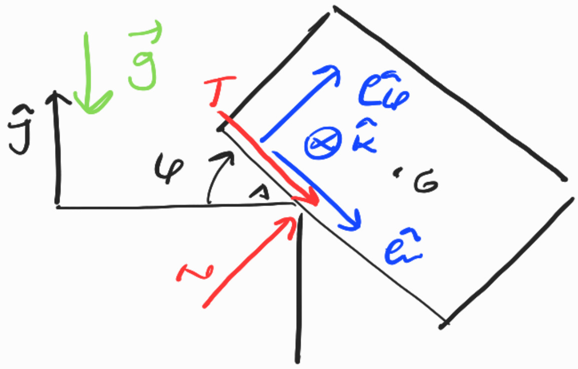
ii) no deslizamiento : $|T| \leq f_E |N|$: $F \leq f_E\, mg$ (I)

iii) no vuelco : $0 \leq x \leq d$: $0 \leq h\,F/mg - (b-d) \leq d$:

$$\boxed{F \geq \left(\frac{b-d}{h}\right) mg} \quad (II)$$

$F \leq \left(\frac{b}{h}\right) mg$ (III) $\longrightarrow$ comparando con (I) $\boxed{F \leq \min\left\{f_E, \frac{b}{h}\right\} mg}$

b) $F = 0$: no se cumple (II), la placa vuelca con respecto a A :



2da Cardinal a la placa desde A :

$$I_A \ddot{\varphi} = (G-A) \times (-mg\,\hat{j}) \cdot \hat{k}$$

$$I_A \overset{(Steiner)}{=} I_G + m(G-A)^2 ,$$

$$I_G = \int_{-a}^{a} dx \int_{-b}^{b} dy \left(\frac{m}{4ab}\right)(x^2+y^2) = \frac{m}{3}(a^2+b^2) \, ;$$

$$G - A = (b-d)\,\hat{e_r} + a\,\hat{e_\varphi} \Rightarrow$$

$$\boxed{\left[\frac{\cancel{m}}{3}(a^2+b^2) + \cancel{m}\left((b-d)^2+a^2\right)\right]\ddot{\varphi} = \cancel{m}g\left[(b-d)\cos\varphi + a\sin\varphi\right]}$$

c) Teor. Cardinal a la placa:

$\hat{e_r}$) $mg\sin\varphi + T = m\,\vec{a}_G \cdot \hat{e_r}$

$\hat{e_\varphi}$) $N - mg\cos\varphi = m\,\vec{a}_G \cdot \hat{e_\varphi}$

$$\vec{r}_G = G - A = (b-d)\,\hat{e_r} + a\,\hat{e_\varphi} \, ;$$

$$\vec{v}_G = -(b-d)\dot{\varphi}\,\hat{e_\varphi} + a\dot{\varphi}\,\hat{e_r}$$

$$\vec{a}_G = -(b-d)\ddot{\varphi}\,\hat{e_\varphi} - (b-d)\dot{\varphi}^2\,\hat{e_r} + a\ddot{\varphi}\,\hat{e_r} - a\dot{\varphi}^2\,\hat{e_\varphi}$$

• $t=0$: $\varphi = 0$, $\dot{\varphi} = 0$ y de la ec. movimiento:

$$\ddot{\varphi}(0) = \left(\frac{1}{3}(4a^2+b^2) + (b-d)^2\right)^{-1} g(b-d)$$

$$\Rightarrow \vec{a}_G(0) = -(b-d)\,\ddot{\varphi}(0)\,\hat{e_\varphi} + a\,\ddot{\varphi}(0)\,\hat{e_r}$$

Luego:

$$\begin{cases} T(0) = ma\,\ddot{\varphi}(0) = \dfrac{(b-d)}{\frac{1}{3}(4a^2+b^2) + (b-d)^2}\, mg \\[2ex] N(0) = mg - m(b-d)\,\ddot{\varphi}(0) = \end{cases}$$

$$= \frac{(4a^2+b^2)/3}{\frac{1}{3}(4a^2+b^2) + (b-d)^2}\, mg \quad (>0 \checkmark)$$

$T(0) \leq f_E N(0)$:
(no deslizamiento)

$$\boxed{f_E \geq 3(b-d)/(4a^2+b^2)}$$

Ej 3) a) $\vec{M}_O^{(ext)} = 0$:

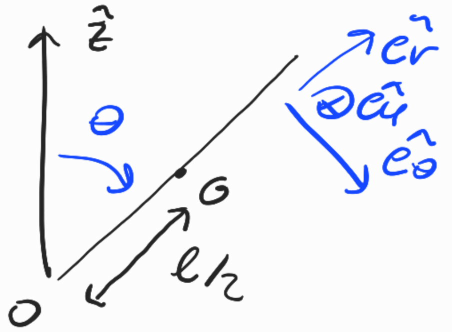
$$P^{(ext)} = \underbrace{\vec{M}_O^{(ext)}}_{=0} \cdot \vec{\omega} + \vec{R}^{(ext)} \cdot \underbrace{\vec{v}_O}_{=0} = 0 : \boxed{\dot{E} = 0}$$

2da Cardinal desde O :

$$\dot{\vec{L}}_O = \vec{M}_O^{(ext)} = \vec{M}_O^{(peso)} = (G-O) \times (-mg\hat{z})$$

sea $L_z = \vec{L}_O \cdot \hat{z}$; $\dot{L}_z = \frac{d}{dt}(\vec{L}_O \cdot \hat{z}) = \dot{\vec{L}}_O \cdot \hat{z}$ $(\dot{\hat{z}} = 0)$

$$\Rightarrow \boxed{\dot{L}_z = \dot{\vec{L}}_O \cdot \hat{z} = (G-O) \times (-mg\hat{z}) \cdot \hat{z} = 0}$$

b) 

$$\mathbb{I}_O\{\hat{e_r}, \hat{e_\theta}, \hat{e_\varphi}\} = I_O \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_O = \int_0^\ell dx \left(\frac{M}{\ell}\right) x^2 = \frac{M\ell^2}{3}$$

$$\vec{L}_O = \mathbb{I}_O \vec{\omega}, \quad \vec{\omega} = \dot{\varphi}\hat{z} + \dot{\theta}\hat{e_\varphi} = \dot{\varphi}(\cos\theta\,\hat{e_r} - \sin\theta\,\hat{e_\theta}) + \dot{\theta}\hat{e_\varphi}$$

$$\vec{L}_O = \dot{\varphi}\cos\theta \underbrace{\mathbb{I}_O\hat{e_r}}_{=0} - \dot{\varphi}\sin\theta \underbrace{\mathbb{I}_O\hat{e_\theta}}_{I_O\hat{e_\theta}} + \dot{\theta} \underbrace{\mathbb{I}_O\hat{e_\varphi}}_{I_O\hat{e_\varphi}}$$

$$= -I_O\dot{\varphi}\sin\theta\,\hat{e_\theta} + I_O\dot{\theta}\,\hat{e_\varphi}$$

$$\Rightarrow L_z = \vec{L}_O \cdot \hat{z} = \boxed{I_O\dot{\varphi}\sin^2\theta} \quad ; \quad L_z(0) = I_O\omega$$

$$E = T + U, \quad T = \frac{1}{2}\vec{\omega} \cdot \mathbb{I}_O\vec{\omega}$$

$$U = Mg\frac{\ell}{2}\cos\theta$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L_O} = \frac{1}{2}\left[\dot{\varphi}(\cos\theta\,\hat{e_r} - \operatorname{sen}\theta\,\hat{e_\theta}) + \dot{\theta}\,\hat{e_\varphi}\right]\cdot$$

$$\left[-I_O\dot{\varphi}\operatorname{sen}\theta\,\hat{e_\theta} + I_O\dot{\theta}\,\hat{e_\varphi}\right]$$

$$= \frac{1}{2}I_O\dot{\varphi}^2\operatorname{sen}^2\theta + \frac{1}{2}I_O\dot{\theta}^2$$

$$\boxed{E = \frac{1}{2}I_O\dot{\theta}^2 + \frac{1}{2}I_O\dot{\varphi}^2\operatorname{sen}^2\theta + Mg\frac{l}{2}\cos\theta}\ ; \quad E(0) = \frac{1}{2}I_O\omega^2$$

$$\Rightarrow \frac{1}{2}I_O\dot{\theta}^2 + \frac{1}{2}I_O\dot{\varphi}^2\operatorname{sen}^2\theta + Mg\frac{l}{2}\cos\theta = \frac{1}{2}I_O\omega^2$$

y de la conservación de L_z tenemos: $\dot{\varphi}\operatorname{sen}^2\theta = \omega$:

$$\frac{1}{2}I_O\dot{\theta}^2 + \frac{1}{2}I_O\frac{\omega^2}{\operatorname{sen}^2\theta} + Mg\frac{l}{2}\cos\theta = \frac{1}{2}I_O\omega^2 \quad \xrightarrow{I_O = Ml^2/3}$$

$$\boxed{\dot{\theta}^2 = \omega^2(1 - \operatorname{sen}^{-2}\theta) - 3g/l\cos\theta} = f(\theta)$$

c) Valores extremos de θ: $\dot{\theta} = 0 \leftrightarrow f(\theta) = 0$:

$$\omega^2\underbrace{(1 - \operatorname{sen}^{-2}\theta)} - 3g/l\cos\theta = 0$$

$$\frac{\operatorname{sen}^2\theta - 1}{\operatorname{sen}^2\theta} = \frac{\cos^2\theta}{\cos^2\theta - 1}$$

$$\left[\omega^2\left(\frac{\cos\theta}{\cos^2\theta - 1}\right) - 3g/l\right]\cos\theta = 0$$

$$\begin{cases} \cos\theta = 0 \ : \ \theta = \theta(0) \\ \quad (\dot{\theta}(0) = 0) \\ \boxed{\omega^2\cos\theta_M - 3g/l(\cos^2\theta_M - 1) = 0} \end{cases}$$

$$(-1 < \cos\theta_M < 0)$$