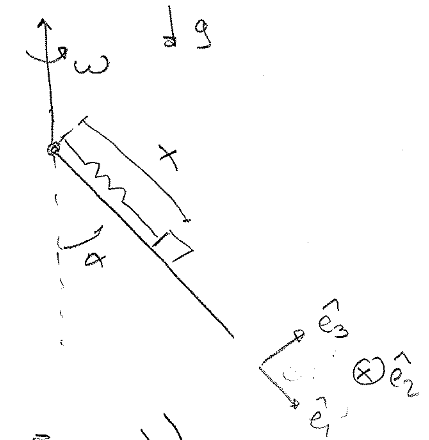
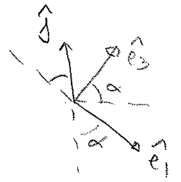
Ejercicio 1: a)





$\hat{j} = -\cos\alpha\, \hat{e}_1 + \sin\alpha\, \hat{e}_3$

$\alpha = \frac{\pi}{4} \Rightarrow \sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}$

$\hat{j} = -\frac{1}{\sqrt{2}}\hat{e}_1 + \frac{1}{\sqrt{2}}\hat{e}_3$

$\vec{r} = x\hat{e}_1 \quad (x \in [0, +\infty))$

$\vec{v} = \dot{x}\hat{e}_1 + x\dot{\hat{e}}_1$

$\vec{\omega} = \omega\hat{j}$

$\dot{\hat{e}}_1 = \vec{\omega}\wedge\hat{e}_1 = (-\cos\alpha\,\hat{e}_1 + \sin\alpha\,\hat{e}_3)\,\omega\wedge\hat{e}_1 = \omega\sin\alpha\,\hat{e}_2$

$\dot{\hat{e}}_2 = \vec{\omega}\wedge\hat{e}_2 = -\omega\cos\alpha\,\hat{e}_3 + \omega\sin\alpha(-\hat{e}_1)$

$\dot{\hat{e}}_3 = \vec{\omega}\wedge\hat{e}_3 = +\omega\cos\alpha\,\hat{e}_2$

$\vec{v} = \dot{x}\hat{e}_1 + x\omega\sin\alpha\,\hat{e}_2$

$\vec{a} = \ddot{x}\hat{e}_1 + \dot{x}\dot{\hat{e}}_1 + \dot{x}\omega\sin\alpha\,\hat{e}_2 + x\omega\sin\alpha\,\dot{\hat{e}}_2$

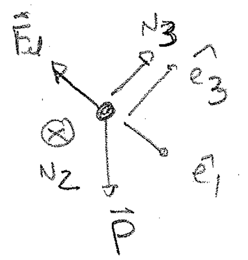
$\vec{a} = (\ddot{x} - x\omega^2\sin^2\alpha)\,\hat{e}_1 + 2\dot{x}\omega\sin\alpha\,\hat{e}_2 - x\omega^2\sin\alpha\cos\alpha\,\hat{e}_3$

Fuerzas

Peso $\quad P = -mg\hat{j} = mg\cos\alpha\,\hat{e}_1 - mg\sin\alpha\,\hat{e}_3$

resorte $\quad \vec{F}_{el} = -kx\,\hat{e}_1$

vínculo $\quad \vec{N} = N_2\hat{e}_2 + N_3\hat{e}_3$



2^{da} Ley

$\hat{e}_1)\ m(\ddot{x} - x\omega^2\sin^2\alpha) = mg\cos\alpha - kx$

$\hat{e}_2)\ 2m\dot{x}\omega\sin\alpha = N_2$

$\hat{e}_3)\ -mx\omega^2\sin\alpha\cos\alpha = N_3 - mg\sin\alpha$

Ec. de mov.

$$\ddot{x} - x\omega^2\sin^2\alpha - g\cos\alpha + \frac{k}{m}x = 0$$

$$\ddot{x} + \left(\frac{k}{m} - \omega^2\sin^2\alpha\right)x - g\cos\alpha = 0$$

b)

puntos de equilibrio: La ec. es $\ddot{x} + f(x) = 0 \Rightarrow \ddot{x} + \frac{dF}{dx} = 0$

busco los mínimos de F

$\hookrightarrow \left.\frac{df}{dx}\right|_{x_{eq}} = 0$

$f(x_{eq}) = 0 \rightarrow x_{eq} = \dfrac{g\cos\alpha}{\frac{k}{m} - \omega^2\sin^2\alpha}$

$\frac{df}{dx} = \frac{k}{m} - \omega^2\sin^2\alpha \Rightarrow$ $\boxed{x_{eq} \text{ es estable si } \frac{k}{m} > \omega^2\sin^2\alpha}$

c) Cond. inicial

$$x(0)=\frac{x_{eq}}{2} \quad y \quad \dot{x}(0)=0$$

Preintegro la ec. de movimiento

$$\frac{1}{2}\left(\dot{x}^2-\dot{x}_0^2\right)+\frac{1}{2}\underbrace{\left(\frac{k}{m}-\omega^2\sin^2\alpha\right)}_{A}\left(x^2-x_0^2\right)-g\cos\alpha\,(x-x_0)=0$$

$\dot{x}_0 = 0$ $\qquad$ $A>0$

$$\dot{x}^2=\dot{x}_0^2+A\left(x_0^2-x^2\right)+2g\cos\alpha\,(x-x_0)$$

Es $\dot{x}^2=g(x)$ $\qquad$ $\dot{x}^2\geq 0 \Rightarrow g(x)\geq 0 \Rightarrow$

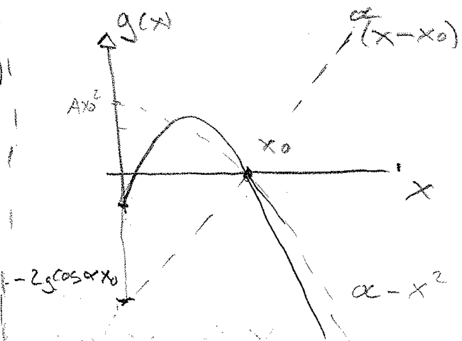
$g(0)=Ax_0^2-2g\cos\alpha \geq 0$ sii $Ax_0^2 \geq 2g\cos\alpha$

si $Ax_0 < 2g\cos\alpha$ entonces no llega a $x=0$

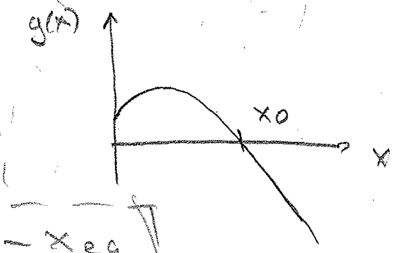
límite $g(0)=0$ si $Ax_0 = 2g\cos\alpha$

$g(x) \xrightarrow[x\to\infty]{} -\infty$ por lo tanto está acotado

graficamente



si $Ax_0 > 2g\cos\alpha$



$$\dot{x}(x_M)=0 \Rightarrow A\underbrace{\left(\frac{x_{eq}^2}{4}-x_M^2\right)}_{\left(\frac{x_{eq}}{2}-x_M\right)\left(\frac{x_{eq}}{2}+x_M\right)}+2g\cos\left(x_M-\frac{x_{eq}}{2}\right)=0$$

$$0=A\left(\frac{x_{eq}}{2}+x_M\right)\left(\frac{x_{eq}}{2}-x_M\right)+2g\cos\alpha\left(x_M-\frac{x_{eq}}{2}\right)$$

$$=\left(\frac{x_{eq}}{2}-x_M\right)\left[A\left(\frac{x_{eq}}{2}+x_M\right)-2g\cos\alpha\right]$$

$\hookrightarrow$ $x_M=\frac{x_{eq}}{2}$ es la posición inicial.

y $A\left(\frac{x_{eq}}{2}+x_M\right)=2g\cos\alpha$

$$x_M=2\frac{g\cos\alpha}{A}-\frac{x_{eq}}{2}$$

$A=\frac{k}{m}-\omega^2\sin^2\alpha$

$x_{eq}=\dfrac{g\cos\alpha}{\frac{k}{m}-\omega^2\sin^2\alpha}$ $\qquad \Rightarrow x_M=2x_{eq}-\frac{1}{2}x_{eq}=\frac{3}{2}x_{eq}$

máximo alejamiento es $\boxed{x_M=\frac{3}{2}x_{eq}}$

$$x_M=\frac{3}{2}\,\frac{g\cos\alpha}{\frac{k}{m}-\omega^2\sin^2\alpha}$$

d) $\vec{v} = \dot{x}\,\hat{e}_1 + x\omega\sin\alpha\,\hat{e}_2$

III

$\vec{v}(x_M) = x_M\omega\sin\alpha\,\hat{e}_2$

$N_2 = 0$ pues $\dot{x}(x_M) = 0$

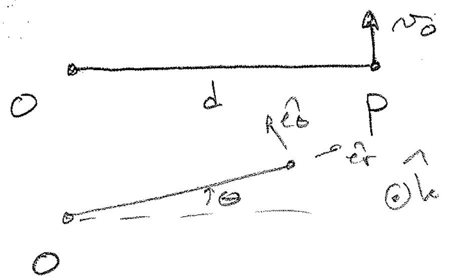
$$N_3 = mg\sin\alpha - m x_M \omega^2 \sin\alpha\cos\alpha$$

$$= mg\sin\alpha - m\frac{3}{2}\frac{g\cos^2\alpha\sin\alpha\,\omega^2}{\frac{k}{m} - \omega^2\sin^2\alpha}$$

$$= \frac{mg}{\sqrt{2}} - \frac{3}{2}\frac{mg}{2}\frac{\omega^2}{\sqrt{2}}\frac{1}{\frac{k}{m} - \frac{\omega^2}{2}} = \frac{\frac{k}{m}\frac{mg}{\sqrt{2}} - mg\omega^2\left(\frac{1}{2\sqrt{2}} + \frac{3}{2}\frac{1}{2\sqrt{2}}\right)}{\frac{k}{m} - \frac{\omega^2}{2}}$$

$$= \frac{\frac{mg}{\sqrt{2}}\left(\frac{k}{m} - \frac{\omega^2}{2}\left(1 + \frac{3}{2}\right)\right)}{\frac{k}{m} - \frac{\omega^2}{2}} = \frac{\frac{mg}{\sqrt{2}}\left(\frac{k}{m} - \frac{5}{4}\omega^2\right)}{\frac{k}{m} - \frac{\omega^2}{2}}$$

12 IV



$\vec{r} = r\,\hat{e}_r$

$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$

a) $\vec{F} = -\frac{k}{r^2}\,\hat{e}_r$ fuerza central e isotrópica

$\Rightarrow$ Es conservativa, $U = -\frac{k}{r}$

$E = \frac{1}{2} m \vec{v}^2 + U(r) \Rightarrow E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{k}{r}$

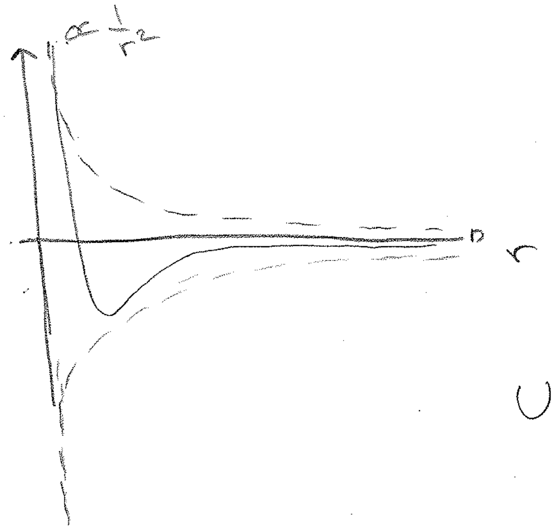
Fuerza central $\Rightarrow \vec{L}_0 = \vec{cte}$ $\quad \vec{L}_0 = \vec{r} \wedge m\vec{v} = m r^2 \dot{\theta}\,\hat{k}$

condición inicial $\vec{r} = d\,\hat{e}_r$, $\vec{v} = v_0\,\hat{e}_\theta$ $\quad \vec{L}_0 = m d v_0\,\hat{k} = l\,\hat{k} \Rightarrow \boxed{l = m d v_0}$

$\Rightarrow E_0 = \frac{1}{2} m v_0^2 - \frac{k}{d}$

$\Rightarrow l = m r^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{l}{m r^2}$

$E_0 = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{l^2}{m} \frac{1}{r^2} - \frac{k}{r}$ $\quad U_{eff}(r) = \frac{1}{2} \frac{l^2}{m} \frac{1}{r^2} - \frac{k}{r}$



$\frac{1}{r} > \frac{1}{r^2}$ cuando $r \to \infty$

$\frac{1}{r^2} > \frac{1}{r}$ cuando $r \to 0$

$U_{eff} \to 0^-$ cuando $r \to \infty$

b) Movimiento circular en el mínimo de U_{eff} $\quad \left.\frac{dU_{eff}}{dr}\right|_{r_c} = 0$

$\frac{dU_{eff}}{dr} = \left.\left(-\frac{l^2}{m} \frac{1}{r^3} + \frac{k}{r^2}\right)\right|_{r_c} = 0 \Rightarrow k r_c = \frac{l^2}{m} \Rightarrow \boxed{r_c = \frac{l^2}{mk}}$

además $r_c \equiv d \Rightarrow d = \frac{l^2}{mk}$ por la cond. inicial $l = m d v_0$

$\Rightarrow d = \frac{m d^2 v_0^2}{k} \Rightarrow \boxed{v_0^2 = \frac{k}{md}}$

$E_0 = U_{eff}(r_c)$ V

$$U_{eff}(r_c) = \frac{1}{2}\frac{\ell^2}{m}\frac{1}{r_c^2} - \frac{k}{r_c}$$

$$= \frac{1}{2}\frac{\ell^2}{m}\frac{1}{\frac{\ell^4}{m^2k^2}} - \frac{k}{\frac{\ell^2}{mk}} = \frac{1}{2}\frac{mk^2}{\ell^2} - \frac{mk^2}{\ell^2} = -\frac{1}{2}\frac{mk^2}{\ell^2}$$

$$= -\frac{1}{2}\frac{mk^2}{m^2d^2v_0^2} = -\frac{1}{2}\frac{k^2}{md^2v_0^2}$$

Para q' d sea el radio de la orbita circular $v_0^2 = \frac{k}{md}$

y $E_0 = -\frac{1}{2}\frac{k^2}{md^2v_0^2} = -\frac{1}{2}\frac{k}{d}$

c) Por el bosquejo si $E_0 < 0$ tengo orbitas acotadas.

$$\Rightarrow E_0 = \frac{1}{2}\frac{\ell^2}{m}\frac{1}{d^2} - \frac{k}{d} = \frac{1}{2}\frac{md^2v_0^2}{d^2} - \frac{k}{d} = \frac{1}{2}mv_0^2 - \frac{k}{d} < 0$$

$$\Rightarrow \frac{1}{2}mv_0^2 < \frac{k}{d} \rightarrow v_0^2 < \frac{2k}{md} \rightarrow 1 < \frac{2k}{mdv_0^2}$$

d) $$E_0 = \frac{1}{2}mv_0^2 - \frac{k}{d} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{m}\frac{1}{r^2} - \frac{k}{r}$$

$$\frac{1}{2}m\dot{r}^2 = \frac{1}{2}mv_0^2 - \frac{k}{d} - \frac{1}{2}\frac{\ell^2}{m}\frac{1}{r^2} + \frac{k}{r}$$

valores extremos $\dot{r} = 0$ $\rightarrow \left(\frac{1}{2}mv_0^2 - \frac{k}{d}\right)r^2 - \frac{1}{2}\frac{\ell^2}{m} + kr = 0$

$\ell^2 = m^2d^2v_0^2$

$$\left(\frac{1}{2}mv_0^2 - \frac{k}{d}\right)r^2 - \frac{1}{2}md^2v_0^2 + kr = 0$$

$$\frac{1}{2}mv_0^2\underbrace{\left(r^2 - d^2\right)}_{(r-d)(r+d)} - k\underbrace{\left(\frac{r^2}{d} - r\right)}_{(r-d)\frac{r}{d}} = 0$$

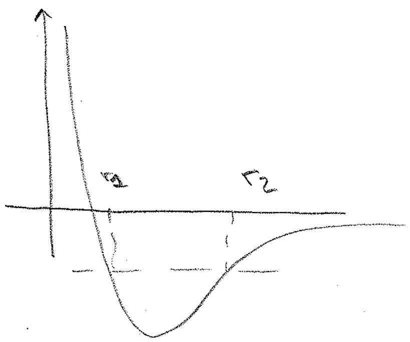
VF

$$\Rightarrow \frac{1}{2} m v_0^2 (r-d)(r+d) - \frac{kr}{d}(r-d) = 0$$

$$\left(\frac{1}{2} m v_0^2 (r+d) - \frac{kr}{d}\right)(r-d) = 0 \rightarrow r_1 = d$$ por condición inicial

$$\left(\frac{1}{2} m v_0^2 - \frac{k}{d}\right) r + \frac{1}{2} m v_0^2 d = 0$$

$$r_2 = -\frac{\frac{1}{2} m v_0^2 d}{\frac{1}{2} m v_0^2 - \frac{k}{d}}$$



observación $\frac{1}{2} m v_0^2 - \frac{k}{d} < 0$

para tener órbitas acotadas

$\Rightarrow r_2 > 0$

$$\frac{1}{2} m v_0^2 < \frac{k}{d}$$

$$r_2 = \frac{1}{2} m v_0^2 d \; \frac{1}{\frac{1}{2} m v_0^2 \left(-1 + \frac{k}{d \frac{1}{2} m v_0^2}\right)}$$

$$r_2 = \frac{d}{\frac{k}{\frac{d}{2} m v_0^2} - 1}$$

para $r_2 > d \Rightarrow \frac{2k}{d\, m v_0^2} - 1 < 1$

$$\hookrightarrow \boxed{\frac{2k}{m v_0^2 d} < 2}$$

para tener órbitas acotadas

$$\Rightarrow \underbrace{\overbrace{1 < \frac{2k}{m v_0^2 d}}\; < 2}$$

para que $r_2 > d$