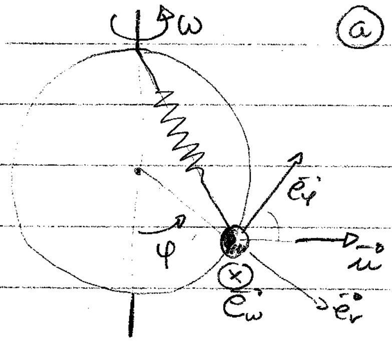
Examen Mecánica Newtoniana

13 febrero 2012

Problema 1 =



ⓐ Emec. no se conserva, pero en sistema relativo $T_r + U + U_{fict.} = c^{te}$

$$\vec{v} = \vec{v}_r + \vec{v}_T = r\dot{\varphi}\,\vec{e}_\varphi + r\,\text{sen}\varphi\,\omega\,\vec{e}_\omega$$

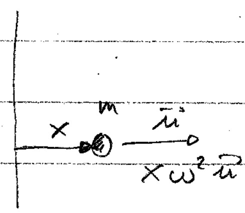
$$T_r = \frac{1}{2} m r^2 \dot{\varphi}^2$$

$$U = U_g + U_e = -mg\,r\cos\varphi + \frac{1}{2}k\,l^2 = kr^2(1+\cos\varphi) - mg\,r\cos\varphi$$

$$l^2 = (r\,\text{sen}\varphi)^2 + (r + r\cos\varphi)^2 = r^2\text{sen}^2\varphi + r^2 + 2r^2\cos\varphi + r^2\cos^2\varphi = 2r^2(1+\cos\varphi)$$

$$\vec{a} = r\ddot{\varphi}\,\vec{e}_\varphi + r\dot{\varphi}\,\dot{\vec{e}}_\varphi + r\cos\varphi\,\omega\dot{\varphi}\,\vec{e}_\omega + r\,\text{sen}\varphi\,\omega\,\dot{\vec{e}}_\omega =$$
$$= r\ddot{\varphi}\,\vec{e}_\varphi - r\dot{\varphi}^2 + 2r\dot{\varphi}\omega\cos\varphi \underbrace{- r\omega^2\,\text{sen}\varphi\,\vec{u}}_{\vec{a}_T}$$

$$-m\,\vec{a}_T = m r \omega^2\,\text{sen}\varphi\,\vec{u} = \vec{F}_{fict}$$



$$\vec{F}_{fict} = m x \omega^2 \vec{u} = -\frac{\partial}{\partial x}\left(-\frac{x^2\omega^2 m}{2}\right) \qquad U_{fict} = \frac{-x^2\omega^2 m}{2}$$

$$(x = r\,\text{sen}\varphi)$$

$$\frac{1}{2} m r^2 \dot{\varphi}^2 + (kr^2 - mgr)\cos\varphi - \frac{m r^2\,\text{sen}^2\varphi\,\omega^2}{2} = c^{te}$$

Ec. mov. $\Rightarrow \dfrac{\partial}{\partial t}\left(T + U + U_{fict.}\right) = 0.$

$$m r^2 \ddot{\varphi} - (kr^2 - mgr)\,\text{sen}\varphi - m r^2\,\text{sen}\varphi\cos\varphi\,\omega^2 = 0.$$

$$\boxed{\ddot{\varphi} - \left[\text{sen}\varphi\cos\varphi\,\omega^2 + \left(k/m - g/r\right)\text{sen}\varphi\right] = 0}$$

(b)

$$U_{ef} = U + U_{fict} = kr^2\cos\varphi - mgr\cos\varphi - \frac{mr^2\,\mathrm{sen}^2\varphi\,\omega^2}{2}$$

$$\frac{\partial U_{ef}}{\partial\varphi} = -kr^2\,\mathrm{sen}\varphi + mgr\,\mathrm{sen}\varphi - mr^2\,\mathrm{sen}\varphi\cos\varphi\,\omega^2 =$$

$$= r\,\mathrm{sen}\varphi\,(mg - kr - mr\omega^2\cos\varphi) = \emptyset.$$

$$\mathrm{sen}\,\varphi = \emptyset \Rightarrow \varphi = \emptyset$$
$$\varphi = \pi$$

$$\cos\varphi_0 = \frac{(mg-kr)}{mr\omega^2} \qquad \exists \Leftrightarrow \quad mg - kr < mr\omega^2$$

$$mg < mr\omega^2 + kr$$

(c) $$\frac{\partial^2 U_{ef}}{\partial\varphi^2} = -kr^2\cos\varphi + mgr\cos\varphi - mr^2\cos^2\varphi\,\omega^2 + mr^2\,\mathrm{sen}^2\varphi\,\omega^2 =$$

$$= (mgr - kr^2)\cos\varphi + mr^2\omega^2 - 2mr^2\omega^2\cos^2\varphi$$

$(\varphi = \emptyset)$ $$\left.\frac{\partial^2 U_{ef}}{\partial\varphi^2}\right|_{\varphi=\emptyset} = (mgr - kr^2) - mr^2\omega^2$$

Si $mg > (kr + mr\omega^2) \Leftrightarrow \varphi = \emptyset$ estable

$(\varphi = \pi)$ $$\left.\frac{\partial^2 U_{ef}}{\partial\varphi^2}\right|_{\varphi=\pi} = -(mgr - kr^2) - mr^2\omega^2$$

Si $kr > mr\omega^2 + mg \Leftrightarrow \varphi = \pi$ estable

$(\varphi = \varphi_0)$ $$\left.\frac{\partial^2 U_{ef}}{\partial\varphi^2}\right|_{\varphi=\varphi_0} = \frac{(mg-kr)^2}{m\omega^2} + mr^2\omega^2 - 2\frac{(mg-kr)^2}{m\omega^2} =$$

$$= -\frac{(mg-kr)^2}{m\omega^2} + mr^2\omega^2$$

Si $(mg - kr)^2 < m^2r^2\omega^4 \Rightarrow (mg - kr) < mr\omega^2$

$mg < (kr + mr\omega^2) \Leftrightarrow \varphi_0$ estable

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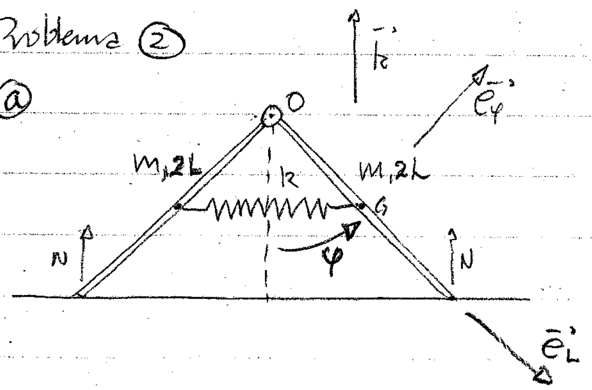
Problema (2)

(a)



$U = U_g + U_e = 2mgL\cos\varphi + 2kL^2\operatorname{sen}^2\varphi$

$U_g = 2mgL\cos\varphi$

$U_e = \frac{k}{2}(2L\operatorname{sen}\varphi)^2$

$$U = 2mgL\cos\varphi + 2kL^2\operatorname{sen}^2\varphi$$

$$\frac{\partial U}{\partial \varphi} = -2mgL\operatorname{sen}\varphi + 4kL^2\operatorname{sen}\varphi\cos\varphi = \emptyset.$$

a- $\operatorname{sen}\varphi = 0 \Rightarrow \varphi = 0 \; ; \; \varphi = \pi$

b- $mg = 2kL\cos\varphi \qquad \cos\varphi = mg/2kL \qquad \exists \Leftrightarrow mg \leqslant 2kL$

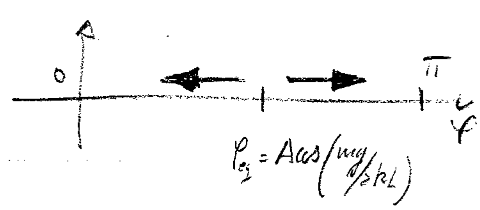
(b) $$\frac{\partial^2 U}{\partial \varphi^2} = -2mgL\cos\varphi + 4kL^2\cos^2\varphi - 4kL^2\operatorname{sen}^2\varphi = $$
$$= -2mgL\cos\varphi - 4kL^2 + 8kL^2\cos^2\varphi$$

$$\left.\frac{\partial^2 U}{\partial \varphi^2}\right|_{mg/2kL = \cos\varphi} = -2mgL \cdot mg/2kL - 4kL^2 + 8kL^2 \cdot \frac{m^2g^2}{4k^2L^2} =$$

$$= -\frac{m^2g^2}{k} - 4kL^2 + 2\frac{m^2g^2}{k} = \frac{m^2g^2}{k} - 4kL^2 < \emptyset$$

si $\exists\, \varphi_{eq}$

El pto de equilibrio $\cos\varphi = mg/2kL$ es inestable.



$E_{mec} = cte = T + U$

$\vec{v}_G = \vec{v}_O + L\dot{\varphi}\vec{e}_\varphi$

$T = \left(\frac{1}{2}m v_G^2\right) \times 2 + \frac{1}{2}\dot{\varphi}^2 I_G$

$\vec{r}_O = 2L\cos\varphi\,\vec{k} \Rightarrow \vec{v}_O = -2L\operatorname{sen}\varphi\,\dot{\varphi}\,\vec{k}$

$I_G = \frac{mL^2}{3}$

$\vec{v}_G = -2L\operatorname{sen}\varphi\,\dot{\varphi}\,\vec{k} + L\dot{\varphi}\,\vec{e}_\varphi$

$$v_G^2 = \vec{v}_G \cdot \vec{v}_G = 4L^2 \operatorname{sen}^2\varphi\, \dot\varphi^2 + L^2\dot\varphi^2 - 4L^2\dot\varphi^2 \operatorname{sen}^2\varphi = L^2\dot\varphi^2$$

$$E_{mec} = \frac{mL^2}{3}\dot\varphi^2 + mL^2\dot\varphi^2 + 2mgL\cos\varphi + 2kL^2\operatorname{sen}^2\varphi = c^{te}$$

$$\frac{\partial E_{mec}}{\partial t} = \phi \Rightarrow \frac{8mL^2}{3}\ddot\varphi - 2mgL\operatorname{sen}\varphi + 4kL^2\operatorname{sen}\varphi\cos\varphi = \phi$$

$$\ddot\varphi = \frac{3}{4}\operatorname{sen}\varphi\left(g/L - 2\frac{k}{m}\cos\varphi\right)$$

$$\ddot\varphi = \frac{3}{2}\frac{k}{m}\operatorname{sen}\varphi\left(\frac{mg}{2kL} - \cos\varphi\right) < \phi$$

Consistente con la inestabilidad del φ_{eq}. $\quad \varphi \in (\phi, \operatorname{Acos} {}^{mg}/_{2kL})$

(c)

$$\vec{a}_G = -2L\cos\varphi\,\dot\varphi^2\,\vec{k} - 2L\operatorname{sen}\varphi\,\ddot\varphi\,\vec{k} + L\ddot\varphi\,\vec{e}_\varphi - L\dot\varphi^2\,\vec{e}_L$$

$$m\vec{a}_G\cdot\vec{k} = -2L\cos\varphi\,\dot\varphi^2 - 2L\operatorname{sen}\varphi\,\ddot\varphi + L\ddot\varphi\operatorname{sen}\varphi + L\dot\varphi^2\cos\varphi = -L\cos\varphi\,\dot\varphi^2 - L\operatorname{sen}\varphi\,\ddot\varphi$$

$$2m\vec{a}_G\cdot\vec{k} = -2mg + 2N$$

$$-mL\left(\cos\varphi\,\dot\varphi^2 + \operatorname{sen}\varphi\,\ddot\varphi\right) = -mg + N$$

$$\dot\varphi_0 = \phi \rightarrow \frac{4mL^2}{3}\dot\varphi^2 + 2mgL\cos\varphi + 2kL^2\operatorname{sen}^2\varphi = U_0$$

$$\dot\varphi^2 = \left[2mgL(\cos\varphi - \cos\varphi_0) + 2kL^2(\operatorname{sen}^2\varphi - \operatorname{sen}^2\varphi_0)\right]\frac{3}{4mL^2}$$

$$\dot\varphi^2 = \frac{3}{2}g/L\left(\cos\varphi - \cos\varphi_0\right) + \frac{3}{2}k/m\left(\operatorname{sen}^2\varphi - \operatorname{sen}^2\varphi_0\right)$$

$$N = mg - mL\cos\varphi\left[3/2\, g/L(\cos\varphi - \cos\varphi_0) + 3/2\, k/m(\operatorname{sen}^2\varphi - \operatorname{sen}^2\varphi_0)\right] - mL\operatorname{sen}\varphi\left[3/4\operatorname{sen}\varphi\left(g/L - 2k/m\cos\varphi\right)\right]$$

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$$N = mg - \frac{3}{2} mg \cos^2\varphi + \frac{3}{2} mg \cos\varphi \cos\varphi_0 - \frac{3}{2} kL \cos\varphi \left(\cos^2\varphi_0 - \cos^2\varphi\right)$$

$$- \frac{3}{4} mg \operatorname{sen}^2\varphi + \frac{3}{2} kL \operatorname{sen}^2\varphi \cos\varphi =$$

$$= mg - \frac{3}{4} mg + \left(\frac{3}{2} mg \cos\varphi_0 - \frac{3}{2} kL \cos^2\varphi_0 + \frac{3}{2} kL\right) \cos\varphi + \left(-\frac{3}{2} mg + \right.$$

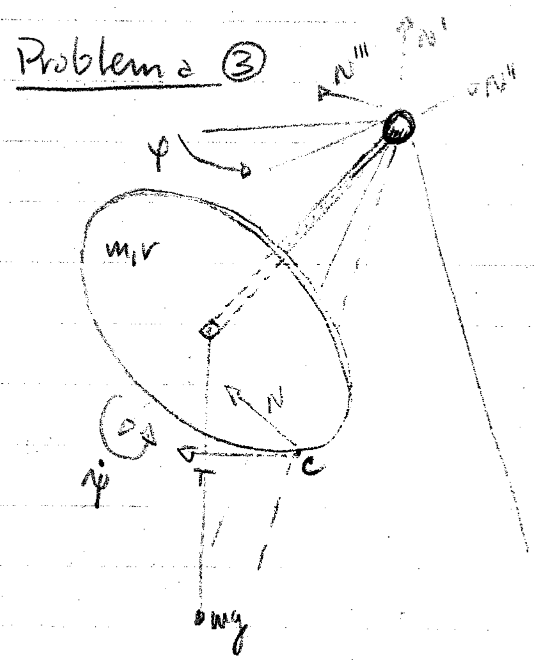
$$\left. + \frac{3}{4} mg\right) \cos^2\varphi + \cancel{\left(\frac{3}{2} kL - \frac{3}{2} kL\right) \cos^3\varphi}$$

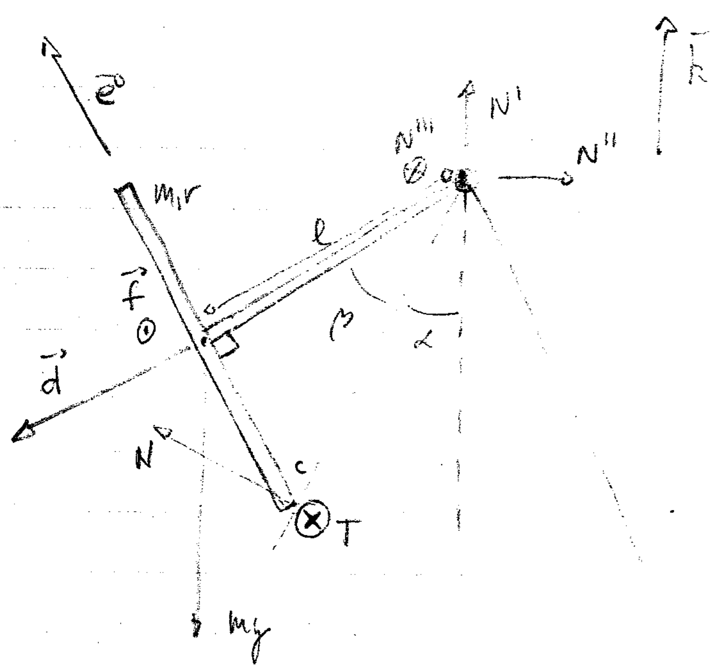
$N(\varphi_d) = \emptyset$ / φ_d = ángulo de desprendimiento

$$\emptyset = \frac{mg}{3} + \underbrace{\left(\frac{3}{2} mg \cos\varphi_0 + \frac{3}{2} kL \operatorname{sen}^2\varphi_0\right)}_{f_0} \cos\varphi_d - \frac{3}{4} mg \cos^2\varphi_d$$

$$\cos\varphi_d = \frac{-f_0 \pm \sqrt{f_0^2 - m^2 g^2}}{-\frac{3}{2} mg}$$

Problema ③





$$\vec{\omega} = \dot\psi \vec{d} + \dot\varphi \vec{k} = \left(\dot\psi - \dot\varphi \cos(\alpha+\beta)\right) \vec{d} + \dot\varphi \operatorname{sen}(\alpha+\beta) \vec{e}^{\,0}$$

$$\mathbb{I}_G = \begin{bmatrix} \frac{mr^2}{4} & & \\ & \frac{mr^2}{4} & \\ & & \frac{mr^2}{2} \end{bmatrix} = \frac{mr^2}{4}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$$

$$\vec f \qquad \vec e \qquad \vec d$$

$$\mathbb{I}_O = \mathbb{I}_G + m\ell^2\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} = \begin{bmatrix} \frac{mr^2}{4} + 3mr^2 & & \\ & \frac{mr^2}{4} + 3mr^2 & \\ & & \frac{mr^2}{2} \end{bmatrix}$$

$$\ell = \frac{r}{tg(\pi/6)} = \sqrt{3}\, r$$

$$\mathbb{I}_O = mr^2\begin{bmatrix} 13/4 & & \\ & 13/4 & \\ & & 1/2 \end{bmatrix}$$

$$\vec M_O^{ext} = mg\ell\, \mathrm{sen}(\alpha+\beta)\, \vec f - \frac{N\ell}{\cos\beta}\vec f + (\ell\vec d - r\vec e)\times\vec T =$$

$$= \left(\frac{3}{2}mgr - 2Nr\right)\vec f + \left(-\sqrt{3}rT\vec e - rT\vec d\right)$$

$$\mathbb{I}_O\vec\omega = mr^2\begin{bmatrix} 13/4 & & \\ & 13/4 & \\ & & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ \dot\varphi\sqrt{3}/2 \\ \dot\psi - \dot\varphi/2 \end{bmatrix} = mr^2\left(\frac{13}{8}\sqrt{3}\dot\varphi\,\vec e + \left(\frac{\dot\psi}{2} - \frac{\dot\varphi}{4}\right)\vec d\right)$$

$$\frac{d\,\mathbb{I}_O\vec\omega}{dt} = mr^2\left(\frac{13\sqrt{3}}{8}\ddot\varphi\,\vec e + \frac{13\sqrt{3}}{8}\dot\varphi\,\dot{\vec e} + \left(\frac{\ddot\psi}{2} - \frac{\ddot\varphi}{4}\right)\vec d + \left(\frac{\dot\psi}{2} - \frac{\dot\varphi}{4}\right)\dot{\vec d}\right)$$

$$\dot{\vec e} = \dot\varphi\vec k\times\vec e = \dot\varphi/2\,\vec f \quad ; \quad \dot{\vec d} = \dot\varphi\vec k\times\vec d = \frac{\sqrt{3}}{2}\dot\varphi\,\vec f$$

$$\frac{d\,\mathbb{I}_O\vec\omega}{dt} = \vec M_O^{ext} \qquad \Rightarrow$$

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$$mr^2\left[\left(\frac{13\sqrt{3}}{8}\ddot{\varphi}\right)\vec{e} + \left(\frac{\ddot{\psi}}{2}-\frac{\ddot{\varphi}}{4}\right)\vec{d} + \left(\frac{13\sqrt{3}}{16}\dot{\varphi}^2 + \frac{\sqrt{3}}{4}\dot{\varphi}\dot{\psi} - \frac{\sqrt{3}}{8}\dot{\varphi}^2\right)\vec{f}\right] =$$

$$= -\sqrt{3}rT\vec{e} - rT\vec{d} + \left(\frac{3}{2}mgr - 2Nr\right)\vec{f}$$

$T = \mu_d N$ (deslizamiento)

$$\left.\begin{aligned} &\frac{13\sqrt{3}}{8}mr^2\ddot{\varphi} = -\sqrt{3}r\mu_d N \\ &\frac{mr^2}{4}\left(2\ddot{\psi}-\ddot{\varphi}\right) = -\mu_d rN \\ &mr^2\left(\frac{11\sqrt{3}}{16}\dot{\varphi}^2 + \frac{\sqrt{3}}{4}\dot{\varphi}\dot{\psi}\right) = \frac{3}{2}mgr - 2Nr \end{aligned}\right\}$$

$$\frac{mr^2}{4}\left(2\ddot{\psi}-\ddot{\varphi}\right) = \frac{13}{8}mr^2\ddot{\varphi} \Rightarrow 2\left(2\ddot{\psi}-\ddot{\varphi}\right) = 13\ddot{\varphi}$$

$$\boxed{4\ddot{\psi} = 15\ddot{\varphi}}$$

$$mr^2\left(\frac{11}{16}\sqrt{3}\dot{\varphi}^2 + \frac{\sqrt{3}}{4}\dot{\varphi}\dot{\psi}\right) = \frac{3}{2}mgr + 2\left(\frac{13}{8\mu_d}mr^2\ddot{\varphi}\right)$$

$$\frac{11\sqrt{3}}{16}\dot{\varphi}^2 + \frac{\sqrt{3}}{4}\dot{\varphi}\dot{\psi} = \frac{3}{2}g/r + \frac{13}{4\mu_d}\ddot{\varphi}$$

$$\boxed{\frac{11\sqrt{3}}{16}\dot{\varphi}^2 + \frac{\sqrt{3}}{4}\dot{\varphi}\dot{\psi} - \frac{13}{4\mu_d}\ddot{\varphi} = \frac{3}{2}g/r}$$

(b)

Rodadura sin deslizamiento $\Rightarrow$ $\vec{v}_c = 0$.
$C \in$ disco

$$\vec{v}_C = \vec{v}_G + \vec{\omega} \times \vec{r}_{GC}$$

$$\vec{r}_G = l\vec{d} \rightarrow \vec{v}_G = l\dot{\vec{d}} = l\frac{\sqrt{3}}{2}\dot{\varphi}\vec{f} = \frac{3}{2}r\dot{\varphi}\vec{f}$$

$$\vec{\omega} \times \vec{r}_{GC} = \left[\frac{\sqrt{3}}{2}\dot{\varphi}\vec{e} + (\dot{\psi} - \dot{\varphi}/2)\vec{d}\right] \times -r\vec{e} =$$

$$= +r(\dot{\psi} - \dot{\varphi}/2)\vec{f}$$

$$\vec{v}_C = \left(\frac{3}{2}r\dot{\varphi} + r\dot{\psi} - \frac{r\dot{\varphi}}{2}\right)\vec{f} = 0 \Rightarrow \boxed{-\dot{\psi} = \dot{\varphi}}$$

r.s.d.

de la ecuación $4\ddot{\psi} = 15\ddot{\varphi} \Rightarrow 4(\dot{\psi} - \dot{\psi}_0) = 15\dot{\varphi}$

$$4\dot{\psi}(t_{rsd}) - 4\dot{\psi}_0 = -15\dot{\psi}(t_{rsd})$$

$$19\dot{\psi}(t_{rsd}) = 4\dot{\psi}_0$$

$$\dot{\psi}(t_{rsd}) = \frac{4}{19}\dot{\psi}_0 = \dot{\psi}(t = \infty)$$

$$\boxed{\vec{\omega}_\infty = \frac{4}{19}\dot{\psi}_0\left(\vec{d} - \vec{k}\right)}$$