Práctico 2: MCD y MCM

$\text{mcd}(a,b) = \text{máx}\{x \in \mathbb{Z} : x|a \text{ y } x|b\}$
$\text{mcm}(a,b) = \text{mín}\{x \in \mathbb{Z}^+ : a|x \text{ y } b|x\}$

Id. Bezout: $\text{mcd}(a,b) = \text{mín}\{s > 0 : s = ax + by \quad x, y \in \mathbb{Z}\}$
Si $\text{mcd}(a,b) = 1 \quad \exists\, x, y$ t.q $\quad 1 = ax + by$
(Obs: x e y no son únicos)

$\text{mcm}(a,b) \cdot \text{mcd}(a,b) = a \cdot b$
$a^* := \dfrac{a}{\text{mcd}(a,b)} \qquad b^* := \dfrac{b}{\text{mcd}(a,b)} \qquad \Rightarrow \quad \text{mcd}(a^*, b^*) = 1$

Ejercicio 3. Demostrar las siguientes afirmaciones:

a. Se define la *sucesión de Fibonacci* como $F_0 = 0$, $F_1 = 1$ y $F_{n+2} = F_{n+1} + F_n$. Demostrar que dos términos consecutivos de la sucesión de Fibonacci son coprimos.

b. Demostrar que $\text{mcd}(7k+3, 12k+5) = 1$ para todo $k \in \mathbb{N}$.

c. Sean $a, b, c, d \in \mathbb{N}$ tales que $(ad - bc)|a$ y $(ad - bc)|c$. Probar que $\text{mcd}(an + b, cn + d) = 1$ para todo $n \in \mathbb{N}$.

(a)
0 1 1 2 3 5 8 13 21 34 ...

Hay que ver que $\text{mcd}(F_n, F_{n+1}) = 1 \quad \forall n \geq 0$

- $\text{mcd}(0,1) = 1$ ✓ $(\text{mcd}(0,a) = a)$
  $\text{mcd}(1,1) = 1$ ...
- Suponemos $\text{mcd}(F_n, F_{n+1}) = 1$. $\Rightarrow \text{mcd}(F_{n+1}, F_{n+2}) = \text{mcd}(F_{n+1}, F_{n+1} + F_n)$

$\rightarrow \text{mcd}(a, b-a) = \text{mcd}(a,b)$

$= \text{mcd}(F_{n+1}, F_{n+1} + F_n - F_{n+1})$
$= \text{mcd}(F_{n+1}, F_n)$
$\overset{H.I}{=} 1$

(b) $\text{mcd}(7K+3, 12K+5) =$

$\text{mcd}(a, b-a) = \text{mcd}(a,b) = \text{mcd}(b, a-b)$

$$\begin{aligned}\operatorname{mcd}(7k+3,\ 12k+5) &= \operatorname{mcd}(7k+3,\ 12k+5-7k-3)\\ &= \operatorname{mcd}(7k+3,\ 5k+2)\\ &= \operatorname{mcd}(5k+2,\ 7k+3-5k-2)\\ &= \operatorname{mcd}(5k+2,\ 2k+1)\\ &= \operatorname{mcd}(2k+1,\ 5k+2-2k-1)\\ &= \operatorname{mcd}(2k+1,\ 3k+1)\\ &= \operatorname{mcd}(3k+1,\ 2k+1-3k-1)\\ &= \operatorname{mcd}(3k+1,\ -k)\\ &= \operatorname{mcd}(k,\ 3k+1-3k)\\ &= \operatorname{mcd}(k,1)\\ &= 1\end{aligned}$$

$\operatorname{mcd}(b, a-bx)$

$x=3$

(c) $(ad-bc) \mid a \Rightarrow \frac{a}{ad-bc} \in \mathbb{Z}$

$(ad-bc) \mid c \Rightarrow \frac{c}{ad-bc} \in \mathbb{Z}$

Queremos ver que $\forall n \geq 0 \quad \operatorname{mcd}(an+b, cn+d) = 1$

$\boxed{n=0: \ \operatorname{mcd}(b,d)=1}$

$\operatorname{mcd}(b,d)=1$ si y sólo si

Bezout: $\exists\, x,y$ t.q. $1 = bx + d\cdot y$

$bc-ad = b\cdot c - ad$

$1 = \frac{bc-ad}{bc-ad}$

$$1 = \frac{ad-bc}{ad-bc} = \underbrace{\frac{a}{ad-bc}}_{x}\cdot d - \underbrace{\frac{c}{ad-bc}}_{y}\cdot b$$

$\operatorname{mcd} = \min\{ s>0 : s = ax+by \quad x,y \in \mathbb{Z}\}$

Si $a2x + b2y = 12 \Rightarrow \operatorname{mcd}(a,b)=1$

$\operatorname{mcd}(a,b) \neq 2$

(c) Bezout: $\exists\, x,y$ tales que

$(an+b)x + (cn+d)y = 1 \iff \operatorname{mcd}(an+b, cn+d) = 1$

$(an+b)\cdot(-c) + (cn+d)\, a = 1 \iff$

$-\cancel{an}c - bc + \cancel{an}c + ad = 1 \iff$

$ad - bc = 1$, no nos sirve, pero dividimos:

$$(an+b)\cdot\frac{-c}{ad-bc} + (cn+d)\frac{a}{ad-bc} = 1 \iff$$

$$\frac{-a\cancel{n}c}{ad-bc} - \frac{bc}{ad-bc} + \frac{ad}{ad-bc} + \frac{c\cancel{n}a}{ad-bc} = 1 \checkmark$$

$a = k\cdot(ad-bc)$ $\quad c = q(ad-bc)$

$\operatorname{mcd}(an+b, cn+d) = \operatorname{mcd}(kn(ad-bc)+b,\ qn(ad-bc)+d)$

$= \operatorname{mcd}(knad - knbc + b,\ qnad - qnbc + d)$

$= \operatorname{mcd}(kn(ad-bc) - qn(ad-bc) + b - d,\ qn(ad-bc)+d)$ ...

$\operatorname{mcd}(a, b-ax) = \operatorname{mcd}(a,b)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} an+b \\ cn+d \end{pmatrix}$$

Ejercicio 4. En cada caso, hallar $a, b \in \mathbb{N}$ que verifiquen las condiciones dadas.

a. $a+b = 122$ y $\operatorname{mcd}(a,b) + \operatorname{mcm}(a,b) = 1802$.

b. $ab = 22275$ y $\operatorname{mcd}(a,b) = 15$.

c. $a+b = 1271$ y $\operatorname{mcm}(a,b) = 330\cdot\operatorname{mcd}(a,b)$.

d. $ab = 1008$ y $\operatorname{mcm}(a,b) = 168$.

(b) $\quad ab = 22275$

$\operatorname{mcd}(a,b) = 15$

$15\cdot\text{mcm} = \operatorname{mcm}(a,b)\cdot\operatorname{mcd}(a,b) = ab = 22275$

$a^* = \dfrac{a}{\text{mcd}}$ $\quad b^* = \dfrac{b}{\text{mcd}}$ $\quad \Rightarrow \quad ab = \text{mcd}^2\cdot a^* b^*$ $\quad \operatorname{mcd}(a^*, b^*) = 1$

Si $c|a$, $c|b$ $\quad \operatorname{mcd}\left(\dfrac{a}{c}, \dfrac{b}{c}\right) = \dfrac{\operatorname{mcd}(a,b)}{c}$ $\quad$ tomando $c = \operatorname{mcd}(a,b)$

$$\boxed{ab = 15^2\cdot a^*\cdot b^* = 22275}$$

$$a^* b^* = \frac{22275}{15^2} = 99 = 3^2 \cdot 11$$

$$\Rightarrow \quad a^* = 3^2 \qquad a = 9 \cdot 15 = 135$$
$$b^* = 11 \qquad b^* = 11 \cdot 15 = 165$$

(a) $a+b = 122$ $\qquad$ $mcd + mcm = 1802$

Tenemos: $\boxed{a \cdot b = mcd \cdot mcm}$

$$a+b = 122 \Rightarrow b = 122 - a$$

$$mcd(a, 122-a) + mcm(a, 122-a) = 1802$$

$$mcd(a, 122) + mcm(a, 122-a) = 1802$$

$$mcd(a, 122.) + \frac{a(122-a)}{mcd(a,122)} = 1802$$

$mcd(a, 122)$: 1, 122, 2, 61

$2 \rightsquigarrow mcd(a,b) = 2 \qquad a = 72 \quad b = 50$

$mcm(a,b) = 1800$

$$mcm = \frac{ab}{mcd} = \frac{a}{mcd} \cdot b$$