Funciones generatrices

$\{a_n\}$ sucesión $\longleftrightarrow$ $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$a_0, a_1, a_2, \ldots$

$$= \sum_{n=0}^{\infty} a_n x^n$$

serie de potencias

polinomio infinito

Ej 1: $\{a_n\} \longmapsto f(x)$ Ej 2 $f(x) \longmapsto a_n$

$x_1 + x_2 + x_3 + x_4 = 12$

$x_i \geq 0$

$$\underbrace{(1 + x + \overset{x^5}{x^2} + \cdots + x^{12})}_{x_1 \quad x_1=5} \underbrace{(1 + \overset{x}{x} + \cdots + x^{12})}_{x_2 \quad x_2=1} \underbrace{(\overset{1}{1} + \cdots + x^{12})}_{x_3 \quad x_3=0} \underbrace{(1 + \cdots + \overset{x^6}{x^{12}})}_{x_4 \quad x_4=6}$$

La cant de soluciones enteras de la ec es el coeficiente de x^{12} que resulta de desarrollar eso.

ⓐ $\underset{a_0}{\underset{\shortparallel}{C_0^6}}, \underset{a_1}{\underset{\shortparallel}{C_1^6}}, \underset{a_2}{\underset{\shortparallel}{C_2^6}}, \underset{a_3}{\underset{\shortparallel}{C_3^6}}, \underset{a_4}{\underset{\shortparallel}{C_4^6}}, C_5^6, \underset{a_6}{\underset{\shortparallel}{C_6^6}}, \underset{a_7}{\underset{\shortparallel}{0}}, \underset{a_8}{\underset{\shortparallel}{0}}, \dots$

$a_n = 0 \quad \forall n \geq 7$

$$f_a(x) = C_0^6 + C_1^6 x + C_2^6 x^2 + C_3^6 x^3 + C_4^6 x^4 + C_5^6 x^5 + C_6^6 x^6 + \cancel{0 x^7 + 0 x^8 + \dots}$$

$$= \sum_{n=0}^{\overset{k}{6}} C_n^6 \overset{a}{x}^n \overset{b}{1}^{6-n} = (x+1)^6$$

$$\boxed{(a+b)^k = \sum_{n=0}^{k} C_n^k a^n b^{k-n}}$$

$$\boxed{f_a(x) = (1+x)^6}$$

ⓑ $C_1^6, 2C_2^6, 3C_3^6, 4C_4^6, 5C_5^6, 6C_6^6, 0, 0, \dots$

$$f_b(x) = 0 + C_1^6 \times 1 + C_2^6\, 2x + C_3^6\, 3x^2 + C_4^6\, 4x^3 + C_5^6\, 5x^4 + C_6^6\, 6x^5$$

$$f_a(x) = C_0^6 + C_1^6 x + C_2^6 x^2 + C_3^6 x^3 + C_4^6 x^4 + C_5^6 x^5 + C_6^6 x^6$$

$$\Longrightarrow f_b(x) = \frac{d}{dx} f_a(x) = \frac{d}{dx}(1+x)^6 = 6(1+x)^5 \Rightarrow \boxed{f_b(x) = 6(1+x)^5}$$

$$a_n = 1 \quad \forall n \qquad 1,1,1,1,\ldots,1,1,\ldots$$

$$f(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$(1-x)(1+x+x^2+x^3+\cdots) = 1 + \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{\cdots} - (\cancel{x} + \cancel{x^2} + \cancel{x^3} + \cancel{\cdots}) = 1$$

$$\Rightarrow (1-x)(1+x+x^2+\cdots) = 1 \Rightarrow 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$\boxed{f(x) = \frac{1}{1-x}}$$

d) $0,0,0,0,1,1,1,1,\ldots,1,\ldots$

$x^0 \quad x \quad x^2 \quad x^3 \quad x^4 \ldots$

$$f(x) = x^4 + x^5 + x^6 + x^7 + \cdots = x^4 \underbrace{(1 + x + x^2 + x^3 + \cdots)}_{= \frac{1}{1-x}} = \frac{x^4}{1-x}$$

(a) $1, -1, 1, -1, 1, -1, 1, -1 \dots$

$\underset{a_0}{\shortparallel}\ \underset{a_1}{\shortparallel}\ \underset{a_2}{\shortparallel} \dots$

$a_n = (-1)^n$ $\qquad (-1)^0 = 1 \quad (-1)^1 = -1 \quad (-1)^2 = 1 \dots$

$$\boxed{a^n b^n = (ab)^n}$$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

$\underset{y}{\shortparallel}$ $y = -x$

$$= \frac{1}{1-(-x)} = \frac{1}{1+x}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \sum_{n=0}^{\infty} \square^n = \frac{1}{1-\square}$$

(b) $1, 0, 1, 0, 1, 0, 1, \dots$

CV $y = x^2$

$$f(x) = 1 + x^2 + x^4 + x^6 + \dots = 1 + \underset{y}{(x^2)} + \underset{y}{(x^2)}^2 + \underset{y}{(x^2)}^3 + \dots = 1 + y + y^2 + y^3 + \dots$$

$x^4 = (x^2)^2 \qquad x^6 = (x^2)^3$

$$= \frac{1}{1-y} = \frac{1}{1-x^2}$$

$$\boxed{f(x) = \frac{1}{1-x^2}}$$

$(x^a)^b = x^{ab}$

(b) $0, 0, 1, a, a^2, a^3, a^4, \ldots \qquad a \in \mathbb{R}$

$$f(x) = x^2 + ax^3 + a^2x^4 + a^3x^5 + a^4x^6 + \cdots = x^2\left(1 + ax + a^2x^2 + a^3x^3 + a^4x^4 + \cdots\right)$$

$$= x^2 \underbrace{\left(1 + ax + (ax)^2 + (ax)^3 + (ax)^4 + \cdots\right)}_{\frac{1}{1-ax}} = \frac{x^2}{1-ax}$$

Ej 2 $\quad f(x) \longmapsto \{a_n\}$

a) $f(x) = (2x - 3)^3$ $\quad$ a $\quad$ b $\quad$ k

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$\boxed{(a+b)^k = \sum_{n=0}^{k} C_n^k a^n b^{k-n}}$$

$$f(x) = \underbrace{C_0^3}_{1}(-3)^3 + \underbrace{C_1^3}_{3} 2x(-3)^2 + \underbrace{C_2^3}_{3}(2x)^2(-3) + \underbrace{C_3^3}_{1}(2x)^3$$

$$= -27 + 54x - 36x^2 + 8x^3 + 0x^4 + 0x^5 + \cdots$$

$a_0 = -27$
$a_1 = 54$
$a_2 = -36$
$a_3 = 8$
$a_n = 0 \quad \forall n \geq 4$

sucesión: $-27, 54, -36, 8, 0, 0, 0 \ldots$

b) $$\frac{1}{1+3x} = \frac{1}{1 - \underbrace{(-3x)}_{y}} = \frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = 1 - 3x + (-3x)^2 + (-3x)^3 + \cdots$$

$$= 1 - 3x + 9x^2 - 27x^3 + \cdots$$

sucesión: $1, -3, 9, -27$

$$a_n = (-3)^n = (-1)^n 3^n$$

(e) $\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\cdot\frac{1}{1-\underbrace{\frac{x}{2}}_{=y}} = \frac{1}{2}\,\frac{1}{1-y} = \frac{1}{2}(1+y+y^2+y^3+\cdots)$

$= \frac{1}{2}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\cdots\right)$

$$\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

$= \frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right) = \frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\cdots$

$\Longrightarrow$ suci $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$ $a_n = 1/2^{n+1}$

(f) $\underbrace{3x^6-9} + \underbrace{\frac{1}{1-x}} = \underbrace{3x^6-9} + \underbrace{1+x+x^2+\cdots+x^5+x^6+\cdots}$

$= 1-9+x+x^2+x^3+x^4+x^5+3x^6+x^6+x^7+\cdots$

$= -8+x+x^2\cdots+x^5+4x^6+x^7+\cdots$

suci $-8, 1, 1, 1, 1, 1, 4, 1, 1, 1. \cdots$

$a_0\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7 \cdots$

$$a_n = -a_{n-1} - b_n$$
$$b_{n+1} = b_n - 3a_{n-1}$$

sumos

$$\Rightarrow a_n + b_{n+1} = -4a_{n-1}$$
$$\Rightarrow b_{n+1} = -a_n - 4a_{n-1}$$
$$\Rightarrow \boxed{b_n = -a_{n-1} - 4a_{n-2}}$$
$n \geq 2$

$$a_n = -a_{n-1} - b_n = -a_{n-1} - (-a_{n-1} - 4a_{n-2})$$
$$\Rightarrow \boxed{a_n = 4a_{n-2}} \quad a_0 = 0 \quad a_1 = -1 \rightsquigarrow a_n$$

$$a_0 = 0 \quad b_0 = 2 \quad b_1 = 1$$

$$a_1 = -a_0 - b_1 = -1$$

$a_0 = 0 \quad c_1 + c_2 = 0$

$a_1 = -1 \quad 2c_1 - 2c_2 = 1$

$\Rightarrow$

$c_1 - c_2 = -1/2$

$c_1 = \frac{-1}{4} \quad c_2 = \frac{+1}{4}$

$$a_n - 4a_{n-2} = 0 \Rightarrow a_n = c_1 2^n + c_2 (-2)^n$$

$$a_n = \frac{1}{4}(-2^n + (-2)^n)$$

$$b_n = -\frac{1}{4}(-2^{n-1} + (-2)^{n-1}) - 1$$

a_n: # Formas de subir los n escalones

$a_n = a_{n-1} + a_{n-2}$

paso

$\lambda_1 \neq \lambda_2$	$a_n = c_1\lambda_1^n + c_2\lambda_2^n$
$\lambda_1 = \lambda_2$	$a_n = c_1\lambda_1^n + c_2 n\lambda_1^n$

Homogenea $a_0=5$ $a_1=12$

$1a_n - 6a_{n-1} + 9a_{n-2} = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$\frac{6 \pm \sqrt{36 - 4 \times 9}}{2} = 3$

$a_n = c_1 3^n + c_2 n 3^n$ Sol general

$a_0=5$ $a_0 = c_1 3^0 + c_2 \cdot 0 \cdot 3^0 = \boxed{c_1 = 5}$

$a_1=12$ $a_1 = c_1 \times 3 + c_2 \times 3 = 12$

$\Rightarrow c_1 + c_2 = 4$ $\boxed{c_2 = -1}$

$a_n = 5 \times 3^n - n3^n$

$a_n = c_1 3^n + c_2 n 3^n$ sol grdl.

$a_0 = 5$: $a_0 = \underbrace{c_1 \times 3^0 + c_2 \times 0 \times 3^0}$ $= c_1 = 5 \Rightarrow c_1 = 5$

$a_1 = 12$: $a_1 = c_1 \times 3^1 + c_2 \times 1 \times 3^1 = 3c_1 + 3c_2 = 12$

$c_1 + c_2 = 4 \Rightarrow 5 + c_2 = 4$

$$\begin{cases} c_1 = 5 \\ 3c_1 + 3c_2 = 12 \end{cases} \Rightarrow \begin{matrix} c_1 = 5 \\ c_2 = -1 \end{matrix}$$

$$\boxed{a_n = 5 \times 3^n - n \times 3^n}$$