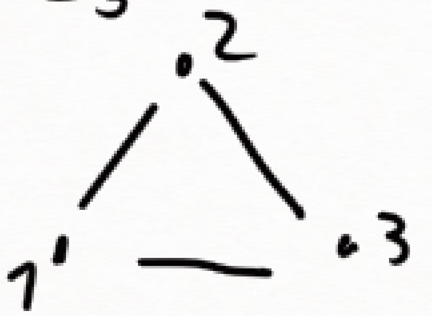
Polinomio cromático

$P(G,\lambda) = P_G(\lambda)$:= cantidad de formas de colorear el grafo disponiendo de λ colores distintos.

Ej: C_3

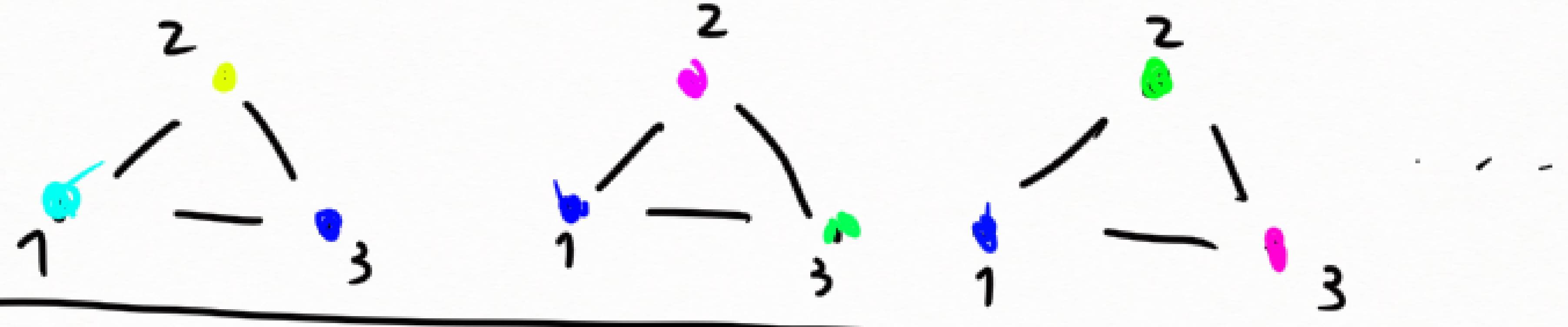


$$P_{C_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)$$

¿Qué es $P_{C_3}(5)$? Cantidad de formas de colorear C_3 disponiendo de 5 colores.

$$P_{C_3}(5) = 5 \cdot 4 \cdot 3 = 60$$

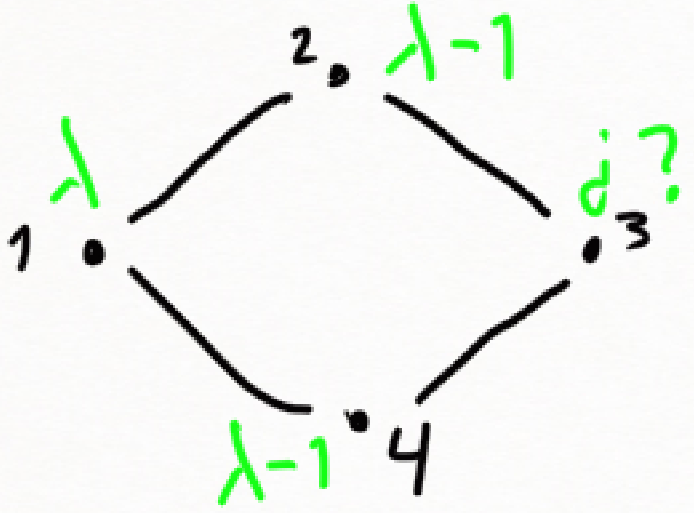
Colores: 

Coloraciones: 

¿$P_G(m) = 0$? Con m colores no alcanza para colorear el grafo

Número cromático $\chi(G) = \min\{\lambda > 0 \ \lambda \in \mathbb{N} \ / \ P_G(\lambda) > 0\}$

C_4



Caso 1

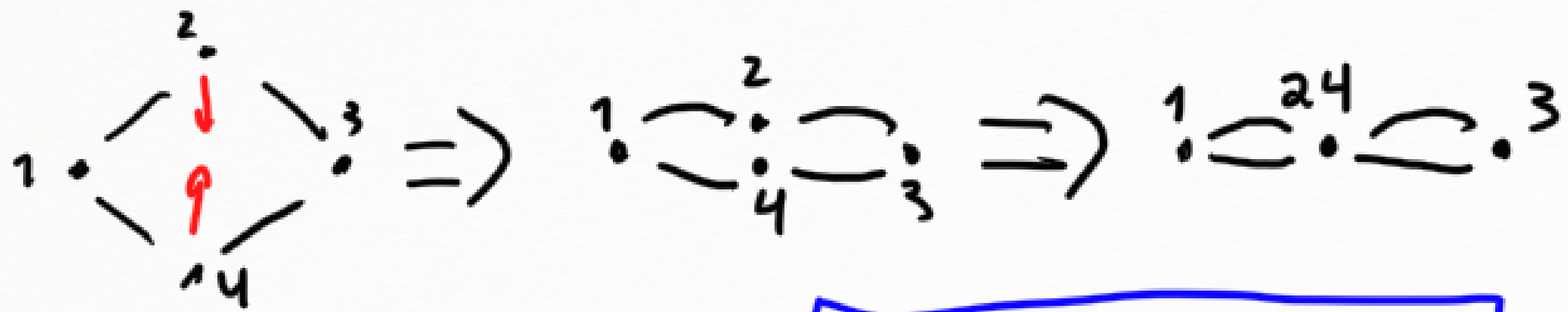
si el 2 y el 4 tiene igual color $\Rightarrow$ Hay $\lambda-1$ disponibles

Caso 2

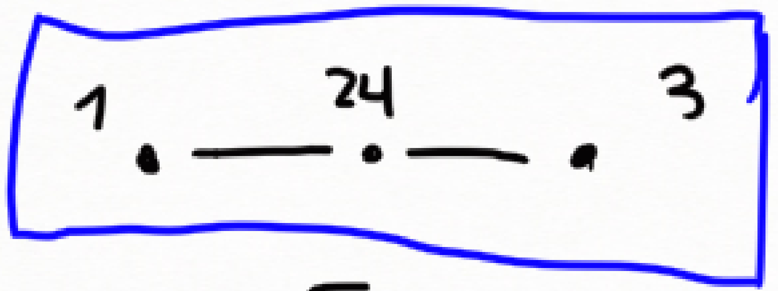
si el 2 y el 4 tienen distinto color $\Rightarrow$ Hay $\lambda-2$ disponibles

Caso 1

El 2 y el 4 tienen el mismo color. Podemos juntarlos sin cambiar el resultado



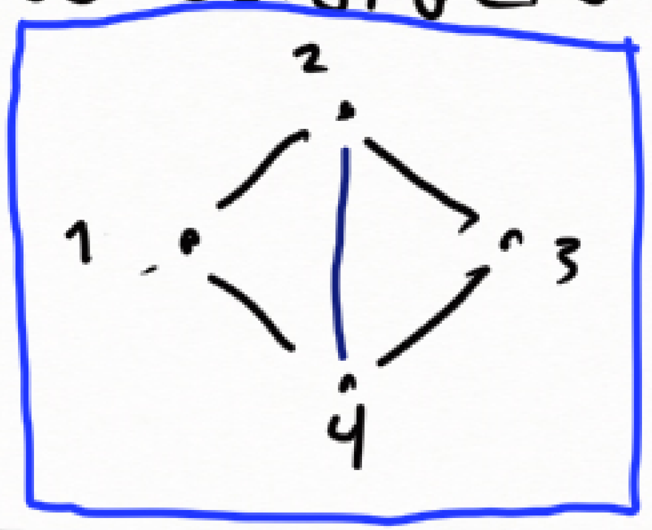
Sacamos aristas múltiples $\Rightarrow$



G_1

Caso 2

El 2 y el 4 tienen distinto color. Podemos unirlos por una arista sin afectar las coloraciones propias.



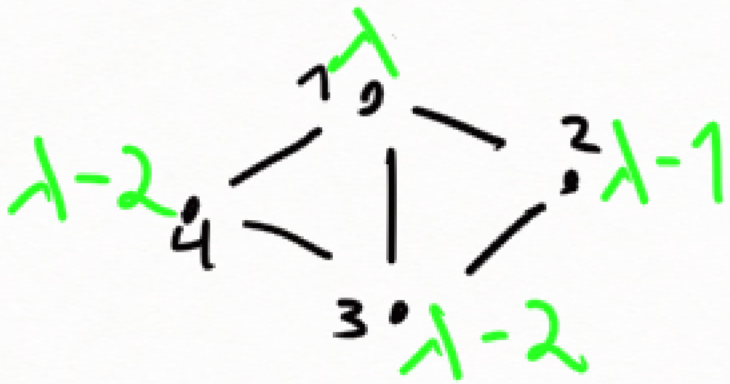
G_2

Por regla de la suma: $\boxed{P_{C_4}(\lambda) = P_{G_1}(\lambda) + P_{G_2}(\lambda) \ \forall \lambda}$

G_1

$P_{G_1}(\lambda) = \lambda(\lambda-1)^2$

G_2



$P_{G_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)^2$

$$\Rightarrow P_{C_4}(\lambda) = \lambda(\lambda-1)^2 + \lambda(\lambda-1)(\lambda-2)^2$$

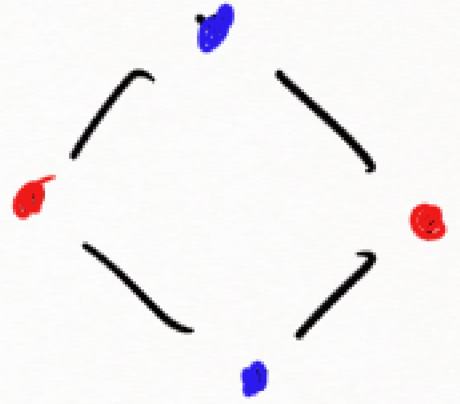
$$P_{C_4}(\lambda) = \lambda(\lambda-1)\left(\lambda-1+(\lambda-2)^2\right)$$

$$= \lambda(\lambda-1)(\lambda^2-3\lambda+3)$$

¿En base a $P_{C_4}(\lambda)$ cuánto vale $\chi(C_4)$?

$$\chi(C_4) = \min\{\lambda > 0 \ \lambda \in \mathbb{N} \ / \ P_{C_4}(\lambda) > 0\}$$

$P_{C_4}(1) = 0$ $\quad$ $P_{C_4}(\underbrace{2}_{\chi(C_4)}) = 2 \times 1 \times (4 \overbrace{-6+3}^{1}) = 2 > 0$



Algoritmo para hallar $\chi(G)$ en base a $\uparrow_G(\lambda)$

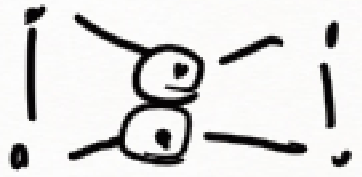
1) arrancamos con $m = 1$

2) Calculamos $\uparrow_G(m)$

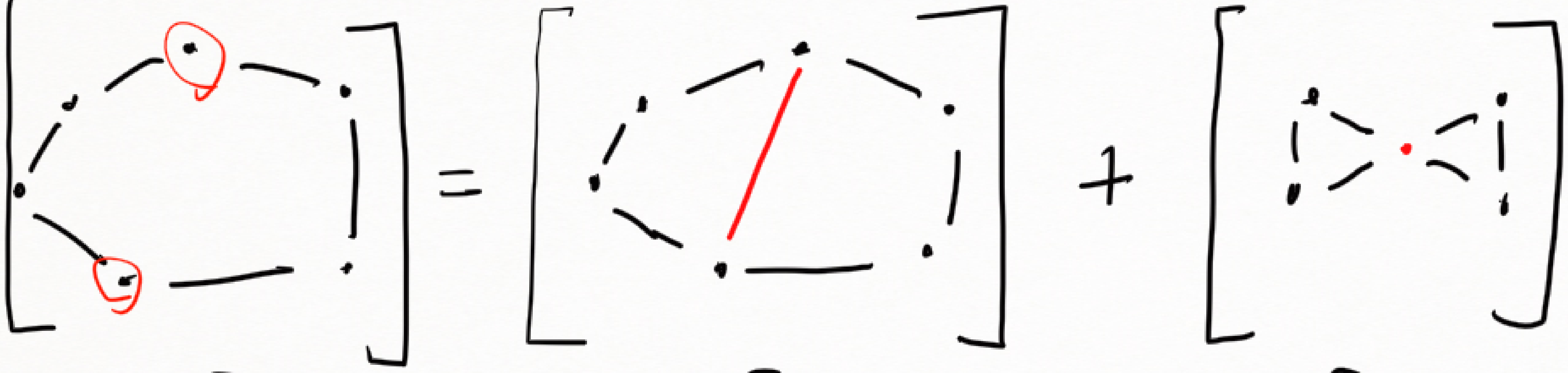
3) si $\uparrow_G(m) > 0 \implies \chi(G) = m$

4) si $\uparrow_G(m) = 0 \implies$ incrementamos m y volvemos al paso 2

cambiamos m por $m+1$



¿Qué pasa si queremos quitar una arista?

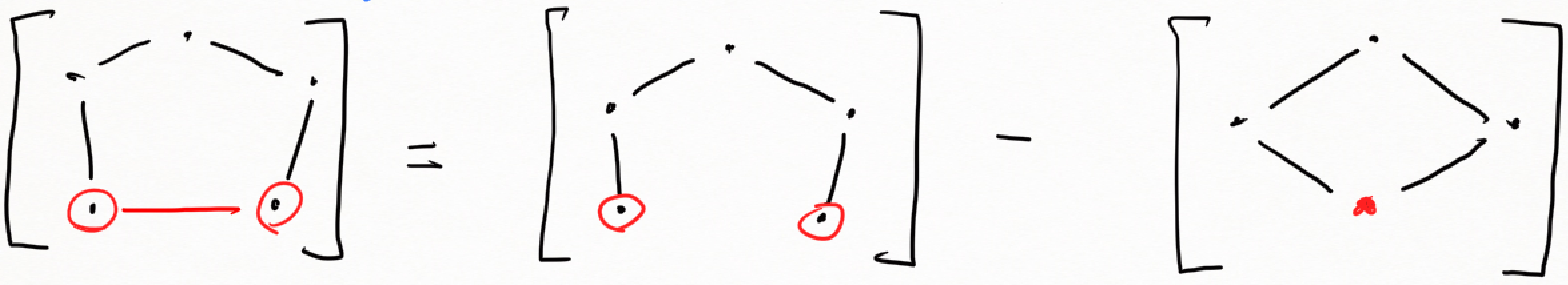


G $\quad G_1 \quad G_2$

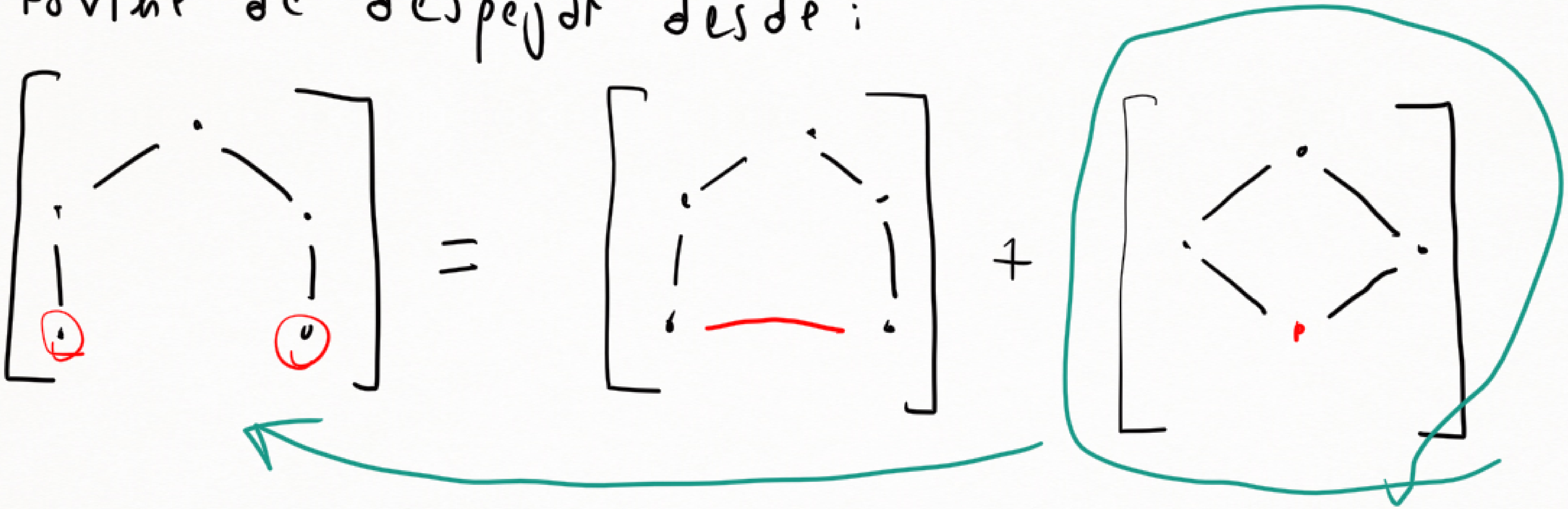
Si despejamos

$$\uparrow_{G_1} = \uparrow_G - \uparrow_{G_2}$$

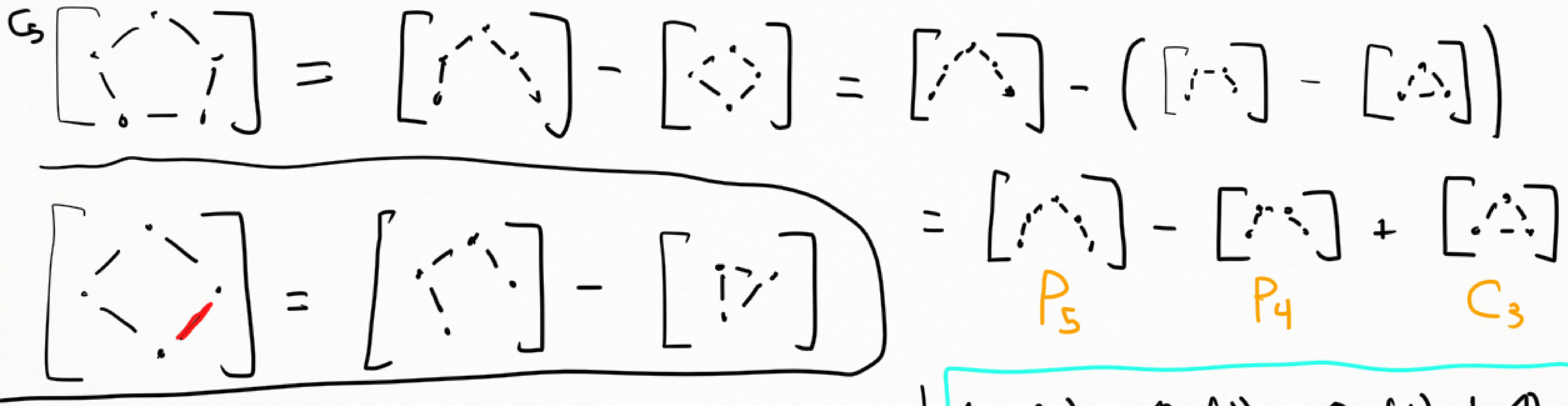
$\boxed{C_5}$

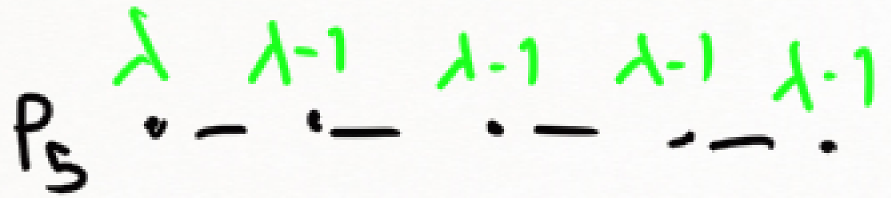


Provine de despejar desde:



paso restando





$\Rightarrow P_{P_5}(\lambda) = \lambda(\lambda-1)^4$

Análogamente

$P_{P_4}(\lambda) = \lambda(\lambda-1)^3$

$P_{C_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)$

$$\boxed{\begin{aligned} P_{C_5}(\lambda) &= P_{P_5}(\lambda) - P_{P_4}(\lambda) + P_{C_3}(\lambda) \\ &= \lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2) \end{aligned}}$$

$$P_{C_5}(\lambda) = \lambda(\lambda-1)^4 - \lambda(\lambda-1)^3 + \lambda(\lambda-1)(\lambda-2)$$

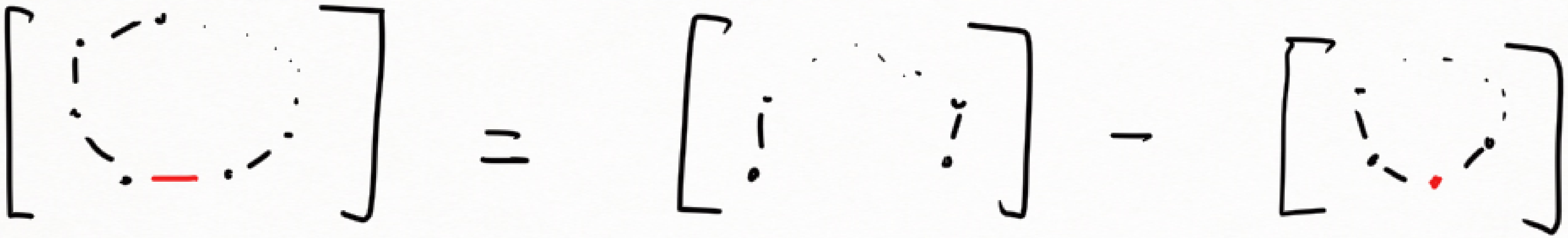
$m=1$; $\quad P_{C_5}(1) = 0 \Rightarrow$ no es 1

$m=2$; $\quad P_{C_5}(2) = 2\times(1)^4 - 2\times(1)^3 + 0 = 0 \Rightarrow$ no es 2

$m=3$; $\quad P_{C_5}(3) = 3\times 2^4 - 3\times 2^3 + 6 = 48 - 24 + 6 = 30$ ✓

$\Rightarrow \chi(C_5) = 3$ calculado con el polinomio cromático.

Caso general C_m



C_m P_m C_{m-1}

Fijamos un λ y definimos una sucesión $c_m = P_{C_m}(\lambda)$

$$P_{C_m}(\lambda) = \underbrace{\lambda(\lambda-1)^{m-1}}_{P_{P_m}(\lambda)} - P_{C_{m-1}}(\lambda) \iff c_m = \lambda(\lambda-1)^{m-1} - c_{m-1}$$

$$\iff \boxed{c_m + c_{m-1} = \lambda(\lambda-1)^{m-1}}$$

- recurrencia

$$c_n + c_{n-1} = \lambda(\lambda-1)^{n-1}$$

Aclaración: Tratamos λ como constante y n como variable

Homogenea: $c_n + c_{n-1} = 0 \Rightarrow c_n^H = k(-1)^n$

Particular: $c_n + c_{n-1} = \lambda(\lambda-1)^{n-1}$ ← como λ cte y n variable exponencial

Probar con $c_n^P = K(\lambda-1)^n$

$$\underbrace{K(\lambda-1)^n}_{(\lambda-1)\cancel{(\lambda-1)^{n-1}}} + K(\lambda\cancel{-1)^{n-1}} = \lambda(\lambda\cancel{-1)^{n-1}}$$

$$K(\lambda-1) + K = \lambda$$

$$K\lambda = \lambda \Rightarrow K = 1$$

$$\Rightarrow c_n^P = (\lambda-1)^n$$

verif

$$\overbrace{(\lambda-1)^n}^{c_n} + \overbrace{(\lambda-1)^{n-1}}^{c_{n-1}} = (\lambda-1)(\lambda-1)^{n-1} + (\lambda-1)^{n-1}$$

$$= (\lambda-1+1)(\lambda-1)^{n-1} = \lambda(\lambda-1)^{n-1} \checkmark$$

$$\boxed{c_n = k(-1)^n + (\lambda-1)^n}$$ → Sol general

$$r_3 = P_{C_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)$$

$$\lambda(\lambda-1)(\lambda-2) = r_3 = K(-1)^3 + (\lambda-1)^3$$

$$\Rightarrow \lambda(\lambda-1)(\lambda-2) = (\lambda-1)^3 - K$$

$$\Rightarrow K = (\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2) = (\lambda-1)\left((\lambda-1)^2 - \lambda(\lambda-2)\right)$$

$$K = (\lambda-1)\left(\cancel{\lambda^2} - \cancel{2\lambda} + 1 - \cancel{\lambda^2} + \cancel{2\lambda}\right) = (\lambda-1)$$

$$\Rightarrow \boxed{r_n = (\lambda-1)(-1)^n + (\lambda-1)^n}$$

$$P_{C_n}(\lambda) = (\lambda-1)(-1)^n + (\lambda-1)^n$$

$$P_{C_n}(2) = (-1)^n + 1 = \begin{cases} 2 & \text{si } n \text{ par} \\ 0 & \text{si } n \text{ impar} \end{cases}$$