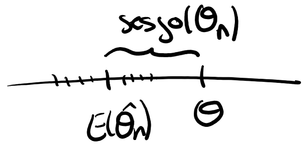
Ejercicio 9
Sean $X_1, \dots, X_n$ $iid \sim \mathcal{U}[0, \theta]$. Interesa estimar el valor de θ.

1. Hallar el estimador de θ por el método de los momentos.
2. Estudiar su sesgo, varianza y error cuadrático medio.

① $X_1, \dots, X_n$ MAS de $X \sim \mathcal{U}[0, \theta]$ $\qquad E(X) = \frac{\theta}{2} \qquad Var(X) = \frac{\theta^2}{12}$

LFGN: $\overline{X_n} \xrightarrow{cs} E(X)$ $\qquad E(X) = \frac{\theta}{2} \Rightarrow 2E(X) = \theta$

estimador de θ: $\hat{\theta}_n = 2\overline{X}_n$



② • $\text{sesgo}(\hat{\theta}_n) = E(\hat{\theta}_n) - \theta$

$$
\begin{aligned}
E(\hat{\theta}_n) &= E(2\overline{X}_n) \\
&= 2\,E\left(\frac{X_1 + \dots + X_n}{n}\right) \\
&= \frac{2}{n} E(X_1 + \dots + X_n) \\
&= \frac{2}{n}\, n\, E(X) \\
&= \frac{\cancel{2}}{\cancel{n}} \cancel{n}\, \frac{\theta}{\cancel{2}} \\
&= \theta
\end{aligned}
$$

$\Rightarrow \text{sesgo}(\hat{\theta}_n) = 0$

•
$$
\begin{aligned}
Var(\hat{\theta}_n) &= Var(2\overline{X}_n) \\
&= 4\, Var\left(\frac{X_1 + \dots + X_n}{n}\right) \\
&= \frac{4}{n^2} Var(X_1 + \dots + X_n) \\
&= \frac{4}{n^2}\, n\, Var(X)
\end{aligned}
$$

$$= \frac{4}{n^2} n \frac{\theta^2}{12}$$

$$= \frac{\theta^2}{3n}$$

- $ECM(\hat{\theta}_n) = E((\hat{\theta}_n - \theta)^2)$

$$= \underbrace{Var(\hat{\theta}_n)}_{E((\hat{\theta}_n - E(\hat{\theta}_n))^2)} + \underbrace{sesgo(\hat{\theta}_n)^2}_{(E(\hat{\theta}_n)-\theta)^2}$$

$$ECM(\hat{\theta}_n) = Var(\hat{\theta}_n) + 0 = \frac{\theta^2}{3n}$$

Ejercicio 4 (5 puntos) Segundo parcial 2019

Sea X el resultado de lanzar un dado trucado cuya f.p.p. es

x	1	2	3	4	5	6
$p(x)$	$\theta/4$	$(1-\theta)/3$	$\theta/4$	$\theta/2$	$(1-\theta)/2$	$(1-\theta)/6$

en donde $0 \leq \theta \leq 1$ es un parámetro. Las siguientes observaciones se obtuvieron al lanzar el dado 10 veces de forma independiente: 1, 1, 5, 1, 4, 4, 5, 4, 2, 2.

Calcular una estimación de θ basada en el método de máxima verosimilitud.

(A) 3/5 **(B)** 1/6 **(C)** 1/2 **(D)** 5/6 **(E)** 2/3 **(F)** 2/5

- función de verosimilitud = probabilidad de observar (1, 1, 5, 1, 4, 4, 5, 4, 2, 2)

$$L(\theta) = P(X_1 = 1) P(X_2 = 1) P(X_3 = 5) \cdots P(X_{10} = 2)$$

$$= \frac{\theta}{4} \cdot \frac{\theta}{4} \cdot \frac{1-\theta}{2} \cdot \frac{\theta}{4} \cdot \frac{\theta}{2} \cdot \frac{\theta}{2} \; \frac{1-\theta}{2} \cdot \frac{\theta}{2} \cdot \frac{1-\theta}{3} \cdot \frac{1-\theta}{3}$$

$$= \left(\frac{\theta}{4}\right)^3 \left(\frac{1-\theta}{2}\right)^2 \left(\frac{\theta}{2}\right)^3 \left(\frac{1-\theta}{3}\right)^2$$

$$= \frac{1}{4^3 2^2 2^3 3^2} \theta^6 (1-\theta)^4$$

- tomamos logaritmo

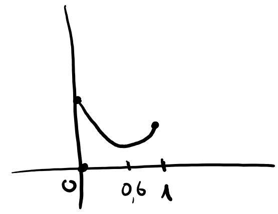
$$h(\theta) = \log(L(\theta))$$

$$= \log\left(\frac{1}{4^3 2^2 2^3 3^2}\right) + \log(\theta^6) + \log((1-\theta)^4)$$

$$= \log\left(\frac{1}{4^3 2^2 2^3 3^2}\right) + 6\log(\theta) + 4\log(1-\theta)$$

$$h'(\theta) = \frac{6}{\theta} - \frac{4}{1-\theta}$$

$$h'(\theta) = 0 \Rightarrow \frac{6}{\theta} - \frac{4}{1-\theta} = 0$$



$$\Rightarrow 6(1-\theta) - 4\theta = 0$$

$$\Rightarrow 6 - 6\theta - 4\theta = 0$$

$$\Rightarrow 6 - 10\theta = 0$$

$$\Rightarrow \boxed{\hat{\theta} = \frac{6}{10}}$$

Ejercicio 5 (5 puntos) Segundo parcial 2019

Un estimador $\hat{\theta}$ de un parámetro θ tiene distribución normal. Se sabe además que

$$\text{ECM}(\hat{\theta}) = 8 \quad \text{y} \quad P(\hat{\theta} \leq \theta) = 0.8413.$$

Hallar el sesgo de $\hat{\theta}$.

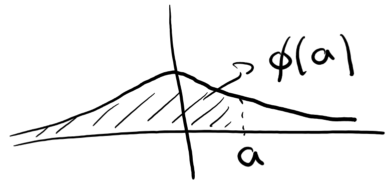
(A) 4 **(B)** $-\sqrt{7}$ **(C)** -2 **(D)** 0 **(E)** $-\sqrt{8}$ **(F)** $2-\theta$

queremos normalizar $\hat{\theta}$

$$Var(\hat{\theta}) = \sigma^2$$

$$sesgo(\hat{\theta}) = E(\hat{\theta}) - \theta \Rightarrow E(\hat{\theta}) = \underbrace{sesgo(\hat{\theta})}_{s} + \theta$$

$$E(\hat{\theta}) = s + \theta$$



$$P(\hat{\theta} \le \theta) = P\left(\frac{\hat{\theta} - (s+\theta)}{\sigma} \le \frac{\theta - (s+\theta)}{\sigma} \right)$$

$$= P\left(\underbrace{\frac{\hat{\theta} - (s+\theta)}{\sigma}}_{\sim \mathcal{N}(0,1)} \le -\frac{s}{\sigma} \right)$$

$$= \phi\left(-\frac{s}{\sigma}\right)$$

$$\Rightarrow \phi\left(-\frac{s}{\sigma}\right) = 0{,}8413 \Rightarrow -\frac{s}{\sigma} = 1 \Rightarrow \begin{cases} -s = \sigma \\ s \le 0 \end{cases} \Rightarrow s^2 = \sigma^2$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	$\boxed{0.8413}$	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

$$ECM(\hat{\theta}) = 8 \Rightarrow Var(\hat{\theta}) + sesgo(\hat{\theta})^2 = 8$$

$$\Rightarrow \sigma^2 + s^2 = 8$$

$$\Rightarrow 2s^2 = 8$$

$$\Rightarrow \begin{cases} s = 2 \\ s = -2 \ \checkmark \end{cases}$$

Segundo parcial 2015

Ejercicio 1. **(28 puntos)** Sea X una variable aleatoria absolutamente continua con densidad dada por

$$f(x,\theta) = \begin{cases} \frac{2x}{\theta} e^{-x^2/\theta} & \text{si } x \geq 0; \\ 0 & \text{si } x < 0, \end{cases}$$

en donde $\theta > 0$ es un parámetro desconocido.

(a) Mostrar que $E(X^2) = \theta$.

(b) Sea $X_1, \ldots, X_n$ una muestra aleatoria simple de X. Hallar $\hat{\theta} = \text{EMV}(\theta)$ el estimador de máxima verosimilitud de θ.

(c) ¿Es $\hat{\theta}$ consistente? Justificar.

(d) Mostrar que $\hat{\theta}$ es insesgado.

a) $$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x,\theta)\, dx$$

$$= \int_0^{+\infty} x^2 \frac{2x}{\theta} e^{-x^2/\theta}\, dx$$

$$f = x^2 \rightsquigarrow f' = 2x$$

$$g' = \frac{2x}{\theta} e^{-x^2/\theta} \rightsquigarrow g = -e^{-x^2/\theta}$$

$$= -x^2 e^{-x^2/\theta} \Big|_0^{+\infty} + \int_0^{+\infty} 2x e^{-x^2/\theta}\, dx$$

$$= \int_0^{+\infty} 2x e^{-x^2/\theta}\, dx \qquad \left(e^{-x^2/\theta}\right)' = -\frac{2x}{\theta} e^{-x^2/\theta}$$

$$= -\theta e^{-x^2/\theta} \Big|_0^{+\infty}$$

$$= \theta$$

$$\Rightarrow E(X^2) = \theta$$

b) buscamos $\hat{\Theta} = EMV(\Theta)$

- función de verosimilitud

$$L(\Theta) = \prod_{i=1}^{n} f(x_i, \Theta)$$

$$= \prod_{i=1}^{n} \frac{2x_i}{\Theta} e^{-x_i^2/\Theta}$$

$$= \frac{2^n}{\Theta^n} e^{-\sum_{i=1}^{n} x_i^2/\Theta} \prod_{i=1}^{n} x_i$$

- tomamos el logaritmo

$$h(\Theta) = \log(L(\Theta))$$

$$= \log\left(\frac{2^n}{\Theta^n}\right) + \log\left(e^{-\sum_{i=1}^{n} x_i^2/\Theta}\right) + \log\left(\prod_{i=1}^{n} x_i\right)$$

$$= \log(2^n) - n\log(\Theta) - \frac{1}{\Theta}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \log(x_i)$$

$$h'(\Theta) = -\frac{n}{\Theta} + \frac{1}{\Theta^2}\sum_{i=1}^{n} x_i^2$$

$$h'(\Theta) = 0 \Rightarrow -\frac{n}{\Theta} + \frac{1}{\Theta^2}\sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow -n\Theta + \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \boxed{\hat{\Theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \overline{X_n^2}}$$

c) ¿ $\hat{\Theta}$ es consistente?

$\hat{\Theta}$ es consistente si $\hat{\Theta} \xrightarrow{cs} \Theta$

$$\begin{cases} \hat{\Theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 = \overline{X_n^2} \\ E(X^2) = \Theta \end{cases}$$

LFGN: $\underbrace{\overline{X_n^2}}_{= \hat{\Theta}} \xrightarrow{cs} \underbrace{E(X^2)}_{= \Theta}$

d) queremos ver que $\hat{\Theta}$ es insesgado

=> queremos ver que $E(\hat{\Theta}) = \Theta$

$$E(\hat{\Theta}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(X_i^2)}_{= \Theta} = \frac{1}{\cancel{n}}\cancel{n}\Theta = \Theta$$