Ejercicio 6

3) $X \sim P(\lambda)$

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{si } k \in \{0,1,2,\dots\}$$

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1$$

$$E(X) = \sum_{k=0}^{\infty} k p_X(k) = \sum_{k=0}^{\infty} \frac{k e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$= \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} \qquad k-1=n$$

$$= \lambda \underbrace{\sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}}_{=1}$$

$$= \lambda$$

$$Var(X) = E(X^2) - E(X)^2$$

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1 \qquad \sum_{k=0}^{\infty} \frac{k e^{-\lambda}\lambda^k}{k!} = \lambda$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P_X(k) = \sum_{k=0}^{\infty} \frac{k^2 e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{k e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(k-1+1) e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(k-1) e^{-\lambda} \lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \lambda \sum_{k=1}^{\infty} \frac{(k-1) e^{-\lambda} \lambda^{k-1}}{(k-1)!} + \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1}}{(k-1)!}$$

$k-1=n$

$$= \lambda \underbrace{\sum_{n=0}^{\infty} \frac{n e^{-\lambda} \lambda^n}{n!}}_{=\lambda} + \lambda \underbrace{\sum_{n=0}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!}}_{=1}$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(5) $X \sim Geo(p)$

$P_X(k) = (1-p)^{k-1} p$ si $k \in \{1, 2, \dots\}$

$$E(X) = \sum_{k=1}^{\infty} k\, p_X(k)$$

$$= \sum_{k=1}^{\infty} k (1-p)^{k-1} p$$

$$= p \underbrace{\sum_{k=1}^{\infty} k (1-p)^{k-1}}_{f(p)}$$

$$\left((1-p)^k\right)' = -k(1-p)^{k-1}$$

Si definimos $F(p) = -\sum_{k=0}^{\infty} (1-p)^k$ tenemos que $F'(p) = f(p)$

$$F(p) = -\sum_{k=0}^{\infty} (1-p)^k = -\frac{1}{1-(1-p)} = -\frac{1}{p}$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

$$f(p) = F'(p) = \frac{1}{p^2}$$

$$E(X) = pf(p) = p\,\frac{1}{p^2} = \frac{1}{p}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 P_X(k)$$

$$((1-p)^k)' = -k(1-p)^{k-1}$$

$$= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

$$(k(1-p)^k)' = -k^2(1-p)^{k-1}$$

$$= p \underbrace{\sum_{k=1}^{\infty} k^2 (1-p)^{k-1}}_{g(p)}$$

Si definimos $G(p) = -\sum_{k=0}^{\infty} k(1-p)^k$ entonces $G'(p) = g(p)$

$$G(p) = -\sum_{k=0}^{\infty} k(1-p)^k = -(1-p)\underbrace{\sum_{k=0}^{\infty} k(1-p)^{k-1}}_{=\frac{1}{p^2}} = -\frac{(1-p)}{p^2}$$

$$= \frac{1}{p} - \frac{1}{p^2}$$

entonces $g(p) = G'(p) = -\frac{1}{p^2} + \frac{2}{p^3}$

$$E(X^2) = p\,g(p) = p\left(\frac{2}{p^3} - \frac{1}{p^2}\right) = \frac{2}{p^2} - \frac{1}{p}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p}$$

Ejercicio 7

Calcular esperanza y varianza de la distribución $\mathrm{BN}(k,p)$ (binomial negativa).

Sugerencia: Utilizar que si $X_1, \ldots, X_k iid \sim \mathrm{Geo}(p)$ entonces $X_1 + \cdots + X_k \sim \mathrm{BN}(k,p)$.

$X \sim BN(k,p)$

$X = X_1 + X_2 + \ldots + X_k$ donde $X_1, \ldots, X_k$ iid $\sim Geo(p)$

$$E(X) = E(X_1 + X_2 + \ldots + X_k) = \underbrace{E(X_1)}_{= 1/p} + \underbrace{E(X_2)}_{= 1/p} + \ldots + \underbrace{E(X_k)}_{= 1/p} = \frac{k}{p}$$

$$Var(X) = Var(X_1 + X_2 + \ldots + X_k) \underset{\uparrow}{=} Var(X_1) + Var(X_2) + \ldots + Var(X_k) = k\left(\frac{1}{p^2} - \frac{1}{p}\right)$$

las X_i son independientes

Ejercicio 8

$X \sim HG(N, D, n)$

$$P_X(k) = \frac{\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

$$E(X) = \sum_{k=0}^{n} k P_X(k) = \sum_{k=0}^{n} \frac{k\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

truco:

truco:

$$\binom{m}{q} = \frac{m!}{q!\,(m-q)!} = \frac{m}{q}\,\frac{(m-1)!}{(q-1)!\,(m-q)!} = \frac{m}{q}\binom{m-1}{q-1}$$

$$\binom{m-1}{q-1} = \frac{(m-1)!}{(q-1)!\,((m-1)-(q-1))!} = \frac{(m-1)!}{(q-1)!\,(m-q)!}$$

$$E(X) = \sum_{k=1}^{n} \frac{k\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

$$= \sum_{k=1}^{n} \frac{k\,\frac{D}{k}\binom{D-1}{k-1}\binom{N-D}{n-k}}{\binom{N}{n}}$$

$$= \sum_{k=1}^{n} \frac{D\binom{D-1}{k-1}\binom{N-D}{n-k}}{\frac{N}{n}\binom{N-1}{n-1}}$$

$$= \frac{Dn}{N} \sum_{k=1}^{n} \underbrace{\frac{\binom{D-1}{k-1}\binom{N-D}{n-k}}{\binom{N-1}{n-1}}}_{\text{queremos que esto sea la fpp de } HG(N-1,\,D-1,\,n-1)}$$

$$(N-1)-(D-1) = N-D \qquad p(i) = \frac{\binom{D-1}{i}\binom{(N-1)-(D-1)}{(n-1)-i}}{\binom{N-1}{n-1}}$$

$$(n-1)-(k-1) = n-k$$

$$= \frac{Dn}{N} \sum_{k=1}^{n} \frac{\binom{D-1}{k-1}\binom{(N-1)-(D-1)}{(n-1)-(k-1)}}{\binom{N-1}{n-1}} \qquad i = k-1$$

$$= \frac{Dn}{N} \underbrace{\sum_{i=0}^{n-1} \frac{\binom{D-1}{i}\binom{(N-1)-(D-1)}{(n-1)-i}}{\binom{N-1}{n-1}}}_{=1}$$

$$E(X) = \frac{Dn}{N}$$

$$Var(X) = \underbrace{\frac{D(D-1)n(n-1)}{N(N-1)} + \frac{Dn}{N}}_{E(X^2)} - \frac{D^2 n^2}{N^2}$$

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= E(X^2 - X + X) - E(X)^2 \\ &= E(X(X-1) + X) - E(X)^2 \\ &= E(X(X-1)) + E(X) - E(X)^2 \end{aligned}$$

$$E(X(X-1)) = \sum \frac{k(k-1)\binom{D}{k}\binom{N-D}{n-k}}{\binom{N}{n}}$$

Ejercicio 9

Se ponen a funcionar en un mismo momento (que tomamos como tiempo 0) dos lamparitas de dos marcas distintas, A y B, que se dejan prendidas hasta que se rompan. Llamemos X al tiempo de duración de la lamparita A e Y al tiempo de duración de la lamparita B. Admitamos que X e Y son independientes, que X sigue una distribución exponencial de parámetro $\lambda_1 > 0$ y que Y sigue una distribución exponencial de parámetro $\lambda_2 > 0$.

Llamemos S al tiempo en que ocurre la primera rotura de alguna de las dos lamparitas y T al tiempo en que se rompe la restante lamparita.

1. Calcular las funciones de distribución de S y T.
2. Calcular $\mathbf{E}(S)$, $\mathbf{E}(T)$.
3. Calcular $\mathbf{E}(ST)$. ¿Son S y T independientes? Justifique la respuesta.
4. Calcular $\mathbf{P}(S = T)$.

$X \sim \exp(\lambda_1)$

$Y \sim \exp(\lambda_2)$

$S = \min\{X, Y\}$

$T = \max\{X, Y\}$

$S + T = X + Y$

$\rightsquigarrow T = X + Y - S$

$E(T) = E(X) + E(Y) - E(S)$