$\gamma:[a,b]\to\mathbb{C}$

Def: Sea γ un camino y $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ continua y $\mathrm{Im}\,\gamma\subset\Omega$

entonces se define
$$\int_\gamma f\,dz=\int_a^b \underbrace{f(\gamma(t))\gamma'(t)dt}_{\in\mathbb{C}}=\int_a^b \mathrm{Re}(f(\gamma(t))\gamma'(t))+i\int_a^b \mathrm{Im}(f(\gamma(t))\gamma'(t))\,dt$$

Teorema: Si $f\in H(\Omega)$ y $\exists\ F\in H(\Omega)$ tal que $f=F'$ entonces $\int_\gamma f(t)dz=0$ para todo camino cerrado.

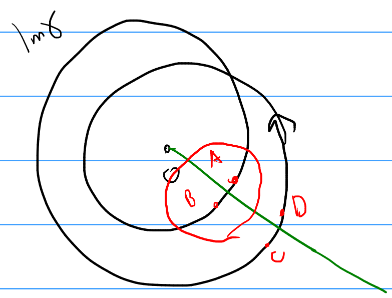
Teorema: Si $f:\Omega\to\mathbb{C}$ continua tal que $\int_\gamma f(t)dz=0\ \forall\gamma$ camino cerrado

entonces $\exists\ F\in H(\Omega)$, $F'=f$. Y de hecho, $\int_\gamma f(t)dz=F(\gamma(b))-F(\gamma(a))$.

Teorema del índice: Sea γ camino cerrado y sea $\mathrm{Ind}_\gamma:\mathbb{C}-\mathrm{Im}\,\gamma\to\mathbb{C}$ con la fórmula:
$$\mathrm{Ind}_\gamma(z)=\frac{1}{2\pi i}\int_\gamma\frac{dw}{w-z}$$

a) $\mathrm{Ind}_\gamma:\mathbb{C}-\mathrm{Im}\,\gamma\to\mathbb{Z}$ b) Ind_γ es continua $\Rightarrow$ es constante en cada componente de $\mathbb{C}-\mathrm{Im}\,\gamma$.

c) Si Ω_1 es la cc no acotada entonces $\mathrm{Ind}_\gamma(z)=0$.



$$\underbrace{\int_\gamma\frac{dw}{w}}=\int_{\gamma_{AC}}+\underbrace{\int_{\gamma_{CD}}}_{\simeq 0}+\int_{\gamma_{DB}}+\underbrace{\int_{\gamma_{BA}}}_{\simeq 0}$$

$(\mathrm{Log}\,w)'=1/w$

$$\simeq(\mathrm{Log}(C)-\mathrm{Log}(A))+\mathrm{Log}(B)-\mathrm{Log}(D)$$

$\mathrm{Log}(B)\simeq\mathrm{Log}(A)+2\pi i$

$$\simeq\underbrace{(\mathrm{Log}(C)-\mathrm{Log}(D))}+(\mathrm{Log}(B)-\mathrm{Log}(A))$$

$$\simeq 2\pi i+2\pi i=\underbrace{2}_{!!}\cdot(2\pi i)$$

Teorema de Cauchy: Sea $f \in H(\Omega)$ y γ camino cerrado tal que $\text{Ind}_\gamma(a) = 0 \ \forall a \notin \Omega$, entonces $\int_\gamma f(z)dz = 0$.

Fórmula de Cauchy: Con las mismas hipótesis se cumple que

$$f(z)\cdot \text{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dz \quad , \text{ y de hecho}$$

$$f^{n}(z)\cdot \text{Ind}_\gamma(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)\,dz}{(w-a)^{n+1}}$$

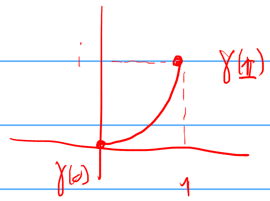
definida de $\mathbb{C} \to \mathbb{C}$.

1) $\int_\gamma e^{z}dz$, $\gamma(t) = (e^t-1)\cdot(e^t-e^\pi)+i\sin t$, $0 \le t \le \pi$

$e^{\gamma(t)}\cdot\gamma'(t)$ luce feo...

$\gamma(0) = 0$, $\gamma(\pi) = 0$ $\Rightarrow$ γ es un camino cerrado, $f \in H(\mathbb{C})$ entonces por el teorema de Cauchy vale 0.

h) $\int_\gamma \sin z\, dz$, $\gamma(t) = t+it^2$, $t \in [0,1]$.



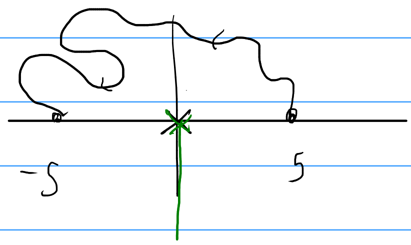
$\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

$\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

$(\cos z)' = i\,\dfrac{(e^{iz}-e^{-iz})}{2} = -\dfrac{1}{i}\,\dfrac{(e^{iz}-e^{-iz})}{2} = -\sin z$.

$$\int_\gamma \sin z\, dz \underset{\text{Tiene primitiva}}{=} -\cos(\gamma(1)) + \cos(\gamma(0)) = -\cos(1+i)+1$$

g) $\int_\gamma \frac{1}{z}\,dz$, γ una curva así:



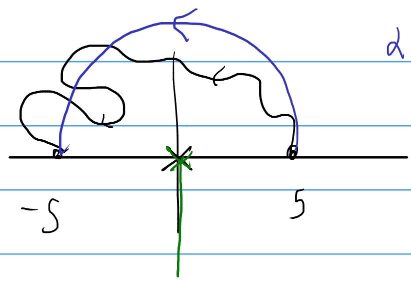
- $\frac{1}{z}: \mathbb{C}-\{0\} \to \mathbb{C}$ no tiene primitiva. Pero puedo usar una primitiva de $\frac{1}{z}: \Omega \to \mathbb{C}$ si $\operatorname{Im}\gamma \subset \Omega$.

Por ejemplo $\Omega = \mathbb{C} - \{z \in \mathbb{C} : Re(z) = 0,\ Im(z) \leq 0\}$

$\Rightarrow$ $Log: \Omega \to \mathbb{C}$, , $Log(w) = \log|w| + i\,Arg_{(-\pi/2, 3\pi/2)}(w)$ es primitiva

$$\int_\gamma \frac{1}{z}\,dz = Log(-5) - Log(5) = i\,Arg_{(-\pi/2,3\pi/2)}(-5) - i\,Arg_{(-\pi/2,3\pi/2)}(5) = i\pi - i0 = i\pi.$$

<u>Otra forma:</u>



$\alpha(t) = 5e^{it}$, $t \in [0,\pi]$

$$\int_\gamma f(z)dz = \int_\alpha f(z)dz$$

ya que

$$\underbrace{\int_{(-\gamma)\cdot(\alpha)}}_{\text{camino cerrado que no da vueltas a } z=0} \frac{1}{z}\,dz \underset{\text{Por Cauchy}}{=} 0 = \int_\alpha f(z)dz - \int_\gamma f(z)dz$$

$$\int_\alpha f(z)dz = \int_0^\pi f(e^{it})\cdot ie^{it}\,dt = \int_0^\pi i\cdot\frac{1}{\cancel{e^{it}}}\cancel{e^{it}}\,dt = i\pi.$$

4a) Probar que no existe una función holomorfa $g: \mathbb{C}-\{0\} \to \mathbb{C}$ y $e^{g(z)} = z$ $\forall z$.

$\left(e^{g(z)}\right)' = 1$

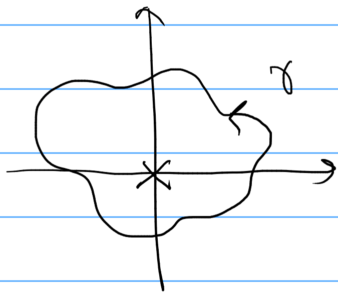
$\parallel$

$\underbrace{e^{g(z)}}_{z} \cdot g'(z) = 1 \Rightarrow g'(z) = 1/z \quad \forall z \in \mathbb{C}-\{0\}$

$\Rightarrow$ $g : \mathbb{C}-\{0\} \to \mathbb{C}$ es ~~una~~ primitiva de $1/z : \mathbb{C}-\{0\} \to \mathbb{C}$

$\Rightarrow$ $\displaystyle\int_\gamma \frac{dz}{z} = g(\gamma(b)) - g(\gamma(a))$ $\forall\, \gamma$ camino con $\mathrm{Im}\,\gamma \subset \mathbb{C}-\{0\}$.

Si



$\displaystyle\int_\gamma \frac{dz}{z} = g(\underbrace{\gamma(b)}) - g(\underbrace{\gamma(a)}) = 0$

$=$

$\parallel$

$2\pi i \, \mathrm{Ind}_\gamma(0) = 2\pi i$ $\lightning$

$\Rightarrow$ No existe dicha g. //