Curso

SISTEMAS Y CONTROL

Clase 13

Fotogramas de los pizarrones de clases filmadas

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$$e^{\lambda t} = 1 + \lambda t + \lambda^2 \frac{t^2}{2!} + \lambda^3 \frac{t^3}{3!} + \dots + \lambda^k \frac{t^k}{k!} + \dots = \sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!}$$

$$e^{At} \triangleq I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots + A^k \frac{t^k}{k!} + \dots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

Si $\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$

con $\left.\begin{matrix} x_0 \\ u(t) \end{matrix}\right\}$ conocidos

$$\Rightarrow \quad x_h(t) = \phi(t,t_0)\,x_0 = e^{A(t-t_0)}x_0 \qquad \text{ya que} \quad \phi(t,t_0) = e^{A(t-t_0)}$$

$$\boxed{x_h(t) = e^{A(t-t_0)}x_0}$$

Cálculo de e^{At} vía Cayley-Hamilton

$$e^{\lambda t}\Big|_{\lambda=\lambda_i} = Q(\lambda)\,d(\lambda)\Big|_{\lambda=\lambda_i} + R_t(\lambda)\Big|_{\lambda=\lambda_i}$$

$$e^{Mt}\Big|_{M=A} = Q(M)\,\underbrace{d(M)}_{0}\Big|_{M=A} + R_t(M)\Big|_{M=A}$$

R_t: grado $n-1$ $\quad R_t(\lambda) = \alpha_0(t)\lambda^{n-1} + \alpha_1(t)\lambda^{n-2} + \dots + \alpha_{n-2}(t)\lambda + \alpha_{n-1}(t)$

$$\Rightarrow \boxed{e^{\lambda_i t} = R_t(\lambda_i) = \alpha_0(t)\lambda_i^{n-1} + \alpha_1(t)\lambda_i^{n-2} + \dots + \alpha_{n-1}(t)}$$

$$\Rightarrow \boxed{e^{At} = R_t(A) = \alpha_0(t)A^{n-1} + \alpha_1(t)A^{n-2} + \dots + \alpha_{n-1}(t)I}$$

Si $d(s) = \det(sI - A)$

$d(\lambda_i) = 0$ (autovalores de A) $\qquad \alpha = V^{-1}\Gamma$

Si λ_i's $\neq$s hay n ecs.

$$\underbrace{\begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \cdots & \lambda_1 & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \cdots & \lambda_2 & 1 \\ \vdots & & & & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \cdots & \lambda_n & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}}_{\alpha} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}}_{\Gamma}$$

Cayley–Hamilton:

$d(A) = 0$

$d(s) = \det(sI - A)$

$$\boxed{A^j = \sum_{i=0}^{n-1} \alpha_{ij} A^i}$$

$A \in \mathbb{C}^{n \times n}$

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \dots + \underbrace{A^k \frac{t^k}{k!}}_{\left(\alpha_{0k} I + \alpha_{1k} A + \dots + \alpha_{n-1\,k} A^{n-1}\right)\frac{t^k}{k!}} + \dots = \overbrace{\left(\sum_{k=0}^{\infty} \alpha_{0k} \frac{t^k}{k!}\right)}^{\alpha_0(t)} I + \overbrace{\left(\Sigma \dots\right)}^{\alpha_1(t)} A + \dots + \overbrace{\left(\Sigma \quad\right)}^{\alpha_{n-1}(t)} A^{n-1}$$

λ_i no son $\neq$s

$$\left.\begin{aligned} e^{\lambda_i t} &= R_t(\lambda_i)\\ t\, e^{\lambda_i t} &= R'_t(\lambda_i)\\ t^2 e^{\lambda_i t} &= R''_t(\lambda_i)\\ &\vdots\\ t^{\mu_i-1} e^{\lambda_i t} &= R^{(\mu_i-1)}(\lambda_i)\end{aligned}\right\} \mu_i \text{ ecs.}$$

Sup. λ_1, con mult. μ_i

$$e^{\lambda t} = Q(\lambda)\, d(\lambda) + R_t(\lambda)$$

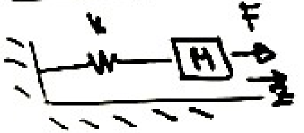
$$\left.\frac{\partial e^{\lambda t}}{\partial \lambda}\right|_{\lambda_i} = \left. t\, e^{\lambda t}\right|_{\lambda_i} = \left.\underbrace{\left[Q'(\lambda)\, d(\lambda) + Q(\lambda)\, d'(\lambda)\right]}_{0}\right|_{\lambda_i} + \left.R'_t(\lambda)\right|_{\lambda_i}$$

$$\frac{\partial^2 e^{\lambda t}}{\partial \lambda^2} = \left. t^2 e^{\lambda t}\right|_{\lambda_i} = \left.\underbrace{\left[Q'' d + Q' d' + Q' d' + Q d''\right]}_{0}\right|_{\lambda_i} + \left.R'_t(\lambda)\right|_{\lambda_i}$$

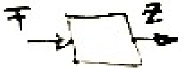
$$d(s) = \overbrace{(s-\lambda_1)\cdots(s-\lambda_1)}\,\overbrace{(s-\lambda_2)(s-\lambda_2)\cdots(s-\lambda_2)}\cdots$$
$$= (s-\lambda_1)^{\mu_1}(s-\lambda_2)^{\mu_2}\cdots(s-\lambda_r)^{\mu_r}$$

$\Rightarrow t\, e$

Ejemplo:



$$\boxed{M\ddot{z} = F - kz}$$



$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} F \end{cases}$$

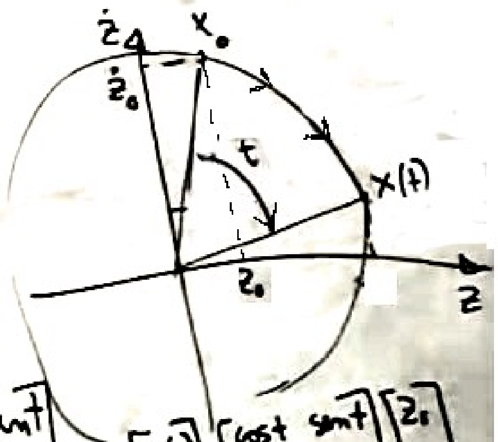
$t_0 = 0$ $\qquad \boxed{F = 0}$

$$x_h(t) = e^{At} \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \left(\frac{k}{M} = 1\right)$$

$$d(s) = \det \begin{bmatrix} s & -1 \\ +1 & s \end{bmatrix} = s^2 + 1 \rightarrow \begin{cases} \lambda_1 = j \\ \lambda_2 = -j \end{cases}$$

C-H

$$\begin{cases} e^{jt} = \alpha_0 j + \alpha_1 \\ e^{-jt} = -\alpha_0 j + \alpha_1 \end{cases} \Rightarrow \begin{cases} \alpha_1(t) = \dfrac{e^{jt} + e^{-jt}}{2} = \cos t \\ \alpha_0(t) = \dfrac{e^{jt} - e^{-jt}}{2j} = \operatorname{sen} t \end{cases}$$



$$e^{At} = \alpha_0 A + \alpha_1 I = \begin{bmatrix} 0 & \operatorname{sen} t \\ -\operatorname{sen} t & 0 \end{bmatrix} + \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} \cos t & \operatorname{sen} t \\ -\operatorname{sen} t & \cos t \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \cos t & \operatorname{sen} t \\ -\operatorname{sen} t & \cos t \end{bmatrix} \begin{bmatrix} z_0 \\ \dot{z}_0 \end{bmatrix}$$

$$e^{\lambda t} = 1 + \lambda t + \lambda^2 \frac{t^2}{2!} + \lambda^3 \frac{t^3}{3!} + \dots + \lambda^k \frac{t^k}{k!} + \dots = \sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!}$$

$$e^{At} \triangleq I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots + A^k \frac{t^k}{k!} + \dots = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

Si $\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases} \Rightarrow \boxed{x_h(t) = e^{A(t-t_0)}\,x_0}$

con $\left.\begin{matrix} x_0 \\ u(t) \end{matrix}\right\}$ conocidos

$x_h(t) = \phi(t,t_0)\,x_0 = e^{A(t-t_0)}\,x_0$ ya que $\phi(t,t_0) = e^{A(t-t_0)}$

$$\mathcal{L}\{e^{At}\} = (sI-A)^{-1} \Rightarrow e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = \left[\quad\right]$$

Ej. $(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ +1 & s \end{bmatrix}^{-1} = \frac{1}{\det(sI-A)} \cdot [\;] = \frac{1}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ -\frac{1}{s^2+1} & \frac{s}{s^2+1} \end{bmatrix}$

$$\mathcal{L}\{e^{At}\} = \frac{1}{d(s)} B(s)$$

$$e^{At} = \left\{ \sum_{i=1}^{r} \left| \sum_{j=0}^{\mu_i - 1} K_{ij}^{k\ell}\, t^j\, e^{\sigma_i t} \operatorname{sen}(\omega_i t + \varphi_{ij}^{k\ell}) \right| \right\}$$

$\lambda_i = \sigma_i + j\,\omega_i$

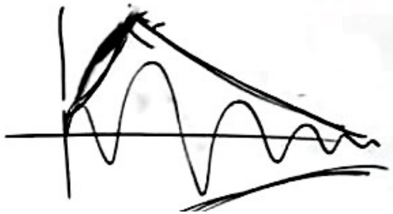
λ_i = autov. de $A \Rightarrow d(\lambda_i) = 0$

$\boxed{\lambda_i = \text{polos del sistema}}$

$t^{\mu} e^{\lambda_i t}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$x_h(t) = e^{At} \cdot x_0$



SISTEMA DINÁMICO LINEAL de PARÁM. CONCENTRADOS — solución general

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases} \quad \text{con} \begin{cases} u(t) \in \mathbb{R}^r \\ y(t) \in \mathbb{R}^m \\ x(t) \in \mathbb{R}^n \end{cases}$$

con $\left.\begin{cases} x(t_0) = x_0 \\ u(t) \ \forall t \geq t_0 \end{cases}\right\}$ conocidas

$$\begin{cases} x(t) = \phi(t,t_0)\,x_0 + \int_{t_0}^{t} \phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma \\ y(t) = \underbrace{C\,\phi(t,t_0)\,x_0}_{y_h(t)} + \underbrace{\int_{t_0}^{t}\left[C(t)\,\phi(t,\sigma)\,B(\sigma) + \delta_{t-\sigma}\,D(\sigma)\right]u(\sigma)\,d\sigma}_{y_\sigma(t)} \end{cases}$$

$$\boxed{x_h(t) = \phi(t,t_0)\,x_0}$$

$$\boxed{y_h(t) = C\,\phi(t,t_0)\,x_0}$$

$$\boxed{y_\sigma = \int_{t_0}^{t} \mathcal{H}(t,\sigma)\,u(\sigma)\,d\sigma}$$

$$\boxed{\mathcal{H}(t,\sigma) = C(t)\,\phi(t,\sigma)\,B(\sigma) + \delta_{t-\sigma}\,D(\sigma)}$$

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

$$x(t) = e^{A(t-t_0)}\,x_0 + \int_{t_0}^{t} e^{A(t-\sigma)}\,B\,u(\sigma)\,d\sigma \qquad \phi(t,t_0) = e^{A(t-t_0)}$$

$$y(t) = \underbrace{C\,e^{A(t-t_0)}\,x_0}_{y_h(t)} + \underbrace{\int_{t_0}^{t}\left[C\,e^{A(t-\sigma)}B + \delta_{t-\sigma}\,D\right]u(\sigma)\,d\sigma}_{y_\sigma(t)}$$

$$y_\sigma(t) = \int_{t_0}^{t} \mathcal{H} \quad u(\sigma)\,d\sigma$$

con $\mathcal{H}(t-\sigma) = \left[C\,e^{A(t-\sigma)}B + \delta_{t-\sigma}\,D\right]$

Caso inv. ent. Tomando tr. de Laplace.

$$\left\{\begin{array}{l} sX(s) - x_0 = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{array}\right. \Rightarrow (sI-A)X(s) = x_0 + BU(s)$$

$$\Rightarrow X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)$$

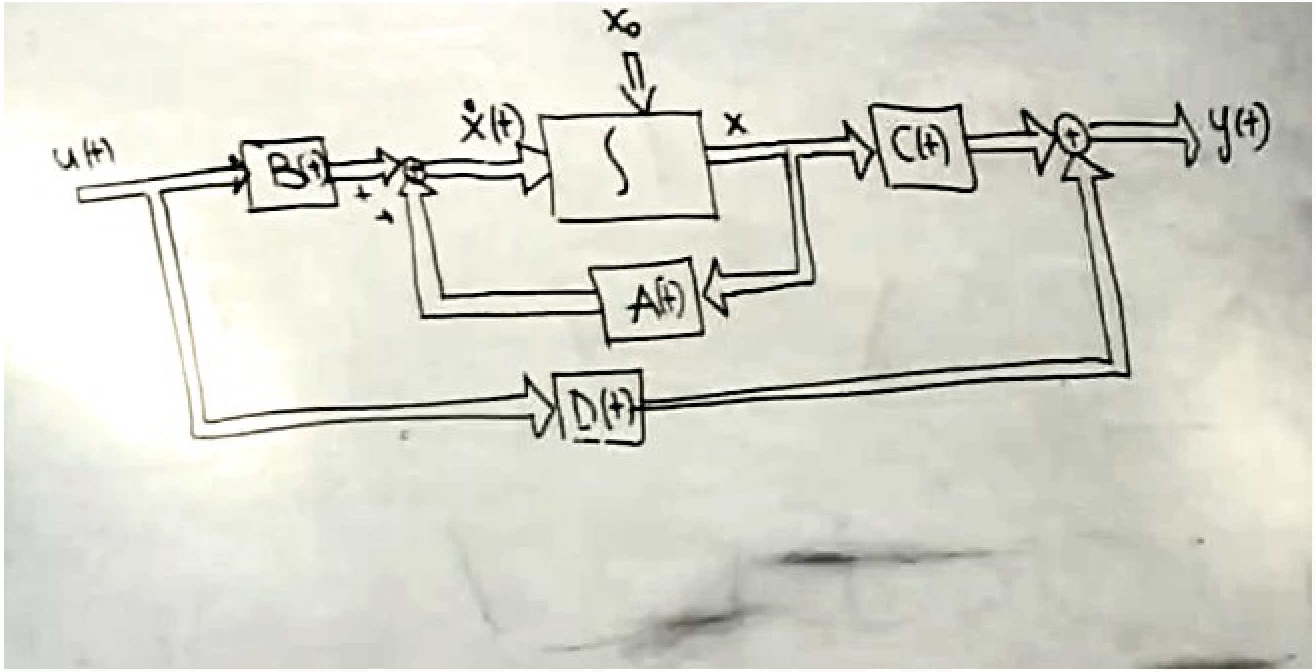
$$Y(s) = C(sI-A)^{-1}x_0 + \left[C(sI-A)^{-1}B + D\right]U(s)$$

$$H(s) = \{h_{ij}(s)\}$$

$$\boxed{Y_\sigma(s) = H(s)\,U(s)}$$

$$\boxed{H(s) = C(sI-A)^{-1}B + D}$$

$$\boxed{H(s) = \mathcal{L}\{\mathcal{H}(t)\}}$$



x_0
$\dot{x}(t)$
x
$u(t)$
$B(t)$
$C(t)$
$y(t)$
$A(t)$
$D(t)$

SISTEMA DINÁMICO LINEAL de PARAM. CONCENTRADOS - Solución general

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases} \quad \text{con} \begin{cases} u(t) \in \mathbb{R}^r \\ y(t) \in \mathbb{R}^m \\ x(t) \in \mathbb{R}^n \end{cases}$$

con $\left\{ \begin{array}{l} x(t_0) = x_0 \\ u(t) \ \forall t \geq t_0 \end{array} \right\}$ conocidas

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

$$\begin{cases} x(t) = \phi(t,t_0)\,x_0 + \int_{t_0}^{t} \phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma \\ y(t) = \underbrace{C\,\phi(t,t_0)\,x_0}_{y_h(t)} + \underbrace{\int_{t_0}^{t} \left[C(t)\,\phi(t,\sigma)\,B(\sigma) + \delta_{t-\sigma}\,D(\sigma)\right] u(\sigma)\,d\sigma}_{y_\sigma(t)} \end{cases}$$

$$\boxed{y_\sigma = \int_{t_0}^{t} \mathcal{H}(t,\sigma)\,u(\sigma)\,d\sigma}$$

$$\boxed{x_h(t) = \phi(t,t_0)\,x_0}$$

$$\boxed{y_h(t) = C\,\phi(t,t_0)\,x_0}$$

$$\boxed{\mathcal{H}(t,\sigma) = C(t)\,\phi(t,\sigma)\,B(\sigma) + \delta_{t-\sigma}\,D(\sigma)}$$

$$x(t) = e^{A(t-t_0)}\,x_0 + \int_{t_0}^{t} e^{A(t-\sigma)}\,B\,u(\sigma)\,d\sigma \qquad \phi(t,t_0) = e^{A(t-t_0)}$$

$$y(t) = \underbrace{C\,e^{A(t-t_0)}\,x_0}_{y_h(t)} + \underbrace{\int_{t_0}^{t} \left[C\,e^{A(t-\sigma)}\,B + \delta_{t-\sigma}\,D\right] u(\sigma)\,d\sigma}_{y_\sigma(t)}$$

$$\boxed{y_\sigma(t) = \int_{t_0}^{t} \mathcal{H} \quad u(\sigma)\,d\sigma}$$

$$\boxed{\text{con } \mathcal{H}(t-\sigma) = \left[C\,e^{A(t-\sigma)}\,B + \delta_{t-\sigma}\,D\right]}$$

Caso inv. ent. Tomando tr. de Laplace.

$$\begin{cases} s\,X(s) - x_0 = A\,X(s) + B\,U(s) \\ Y(s) = C\,X(s) + D\,U(s) \end{cases} \quad \Rightarrow (sI-A)\,X(s) = x_0 + B\,U(s)$$

$$\Rightarrow X(s) = \underbrace{(sI-A)^{-1}}\,x_0 + (sI-A)^{-1}B\,U(s)$$

$$Y(s) = \underbrace{C(sI-A)^{-1}x_0}_{Y_h(s)} + \underbrace{\underbrace{\left[C(sI-A)^{-1}B + D\right]}_{H(s)} U(s)}_{Y_\sigma(s)}$$

$$\boxed{Y_\sigma(s) = H(s)\,U(s)}$$

$$\boxed{H(s) = C(sI-A)^{-1}B + D}$$

$$\boxed{H(s) = \mathcal{L}\{\mathcal{H}(t)\}}$$