Curso

SISTEMAS Y CONTROL

Clase 11

Fotogramas de los pizarrones de clases filmadas

Prof. Rafael Canetti

Instituto de Ingeniería Eléctrica,
Facultad de Ingeniería, Universidad de la República
Montevideo, Uruguay.
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SISTEMA DINÁMICO LINEAL de PARÁMETROS CONCENTRADOS

Solución general

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases}$$

con $\begin{cases} u(t) \in \mathbb{R}^r \\ x(t) \in \mathbb{R}^m \quad \forall t \geq t_0 \\ y(t) \in \mathbb{R}^q \end{cases}$

$\left.\begin{matrix} u(t) \ \forall t \geq t_0 \\ x(t_0) = x_0 \end{matrix}\right\}$ conocidas

¿$x(t)$, $y(t)$?

Hipótesis $A(t), B(t), C(t), D(t), u(t)$ secc. continuas en t

Como la rep. es lineal, se sabe que si $x(t) = \varphi(t, u, x_0, t_0)$

$$\Rightarrow x(t) = \underbrace{\varphi(t, \underline{0}, x_0, t_0)}_{x_h(t)} + \underbrace{\varphi(t, u, \underline{0}, t_0)}_{x_f(t)}$$

$$\dot{x}_h(t) = A(t)\,x_h(t)$$

con $x_h(t_0) = x_0$

$$\boxed{x(t) = \Phi(t, t_0)\,x_0}$$

$$\Phi(t, t_0) = \left[\ \varphi(t, \underline{0}, e_1, t_0) \ \middle|\ \varphi(t, \underline{0}, e_2, t_0) \ \middle|\ \cdots \ \middle|\ \varphi(t, \underline{0}, e_n, t_0)\ \right]$$

1) $\Phi(t_0, t_0) = I$

2) $\Phi(t, t_0)$ es la sol. única de la ec. dif. mat $\boxed{\dfrac{\partial \Phi(t, t_0)}{\partial t} = A(t)\,\Phi(t, t_0)}$ con la cond. inic. $\Phi(t_0, t_0) = I$

props:

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\,\Phi(t_1, t_0) \quad ;$$

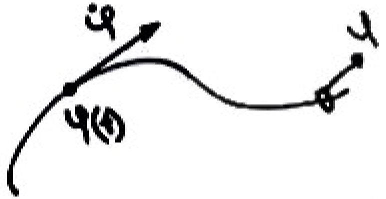
$x_2 = \Phi(t_2, t_0)\,x_0$

$x_2 = \Phi(t_2, t_1)\,x_1 = \Phi(t_2, t_1)\,\Phi(t_1, t_0)\,x_0$

Picard

$$\text{Sea} \begin{cases} \dot{\varphi}(t) = f(\varphi(t), t) \\ \text{con } \varphi(t_0) = \varphi_0 \end{cases} \quad \varphi(t) \in \mathbb{R}^n \ \forall t \geq t_0$$

$$\text{Si } f(\cdot,\cdot) \text{ es} \begin{cases} \text{secc. cont. en } t \\ \text{Lipschitz en } \varphi \end{cases} \Rightarrow \begin{array}{l} 1) \ \underline{\exists \text{ sol } \varphi(t) \text{ por } \varphi_0} \\ 2) \text{ es única} \\ 3) \text{ es continua} \end{array}$$



PROPOSICION

$$\boxed{x(t) = \underbrace{\phi(t,t_0)x_0}_{x_h(t)} + \underbrace{\int_{t_0}^{t} \phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma}_{x_\sigma(t)}}$$

$$\frac{\partial}{\partial t}\left(\int_0^t \underbrace{f(\sigma)}\,d\sigma\right) = f(\sigma)\Big|_{\sigma=t}$$

Demostración:

$$\dot{x}(t) = \underbrace{\dot{\phi}(t,t_0)}_{A(t)\phi(t,t_0)}x_0 + \underbrace{\phi(t,t)}_{I}B(t)u(t) + \int_{t_0}^{t} \overbrace{\dot{\phi}(t,\sigma)}^{A(t)\phi(t,\sigma)}B(\sigma)u(\sigma)\,d\sigma = A(t)\,\phi(t,t_0)x_0 + B(t)u(t) + \int_{t_0}^{t} A(t)\,\phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma$$

$$\frac{\partial}{\partial t}\int_0^t f(t,\sigma)\,d\sigma = f(\sigma)\Big|_{\sigma=t} + \int_0^t \frac{\partial f(t,\sigma)}{\partial t}\,d\sigma$$

$$\boxed{\dot{x}(t) = A(t)\underbrace{\left[\phi(t,t_0)x_0 + \int_{t_0}^{t}\phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma\right]}_{x(t)} + B(t)u(t)}$$

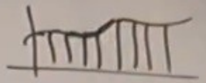
satisface la E.D.

$$x(t_0) = \underbrace{\phi(t_0,t_0)}_{I}x_0 + \underbrace{\int_{t_0}^{t_0}\phi(t_0,\sigma)B(\sigma)u(\sigma)\,d\sigma}_{0} = x_0$$

PROPOSICIÓN

$$x(t) = \underbrace{\phi(t,t_0)x_0}_{x_h(t)} + \underbrace{\int_{t_0}^{t} \phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma}_{x_\sigma(t)}$$

$$y(t) = \underbrace{C(t)\phi(t,t_0)x_0}_{y_h(t)} + \underbrace{C(t)\int_{t_0}^{t} \phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma + D(t)u(t)}_{y_\sigma(t)}$$



$$y(t) = \underbrace{C(t)\phi(t,t_0)x_0}_{y_h(t)} + \underbrace{\int_{t_0}^{t} \left[C(t)\phi(t,\sigma)B(\sigma) + \delta_{t-\sigma} D(\sigma)\right] u(\sigma)\,d\sigma}_{y_\sigma(t)}$$

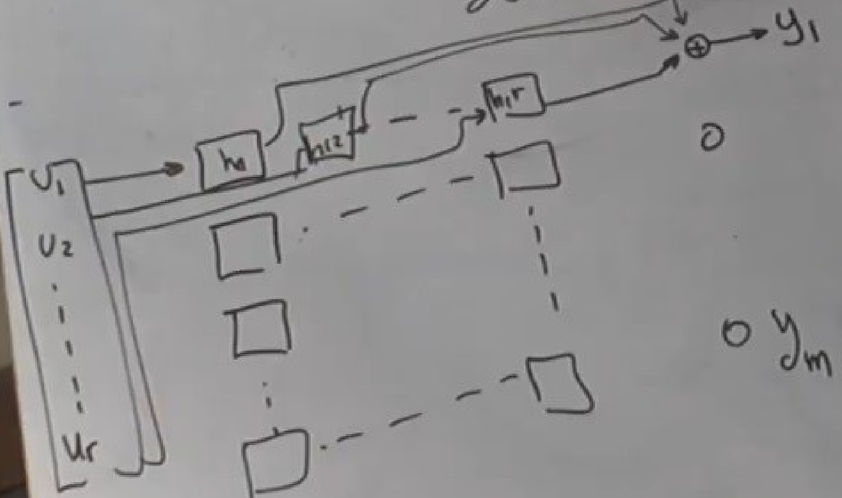
$$y_\sigma(t) = \int_{t_0}^{t} \mathcal{H}(t,\sigma)\,u(\sigma)\,d\sigma$$

$\mathcal{H}(t,\sigma)$: matriz de respuesta a impulso

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_i(t) \\ y_m(t) \end{bmatrix} = \int_{t_0}^{t} \underbrace{\begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} h_{11}(t,\sigma) & \cdots & h_{r1}(t,\sigma) \\ \vdots & h_{ij}(t,\sigma) & \vdots \\ h_{m1}(t,\sigma) & \cdots & h_{mr}(t,\sigma) \end{bmatrix} \end{array}}_{\mathcal{H}(t,\sigma)} \begin{bmatrix} u_1(\sigma) \\ u_2(\sigma) \\ \vdots \\ u_j(\sigma) \\ \vdots \\ u_r(\sigma) \end{bmatrix} d\sigma$$

$$u_j(\sigma) = \delta_\tau$$



$$\boxed{x(t) = \underbrace{\phi(t,t_0)x_0}_{x_h(t)} + \underbrace{\int_{t_0}^{t} \phi(t,\sigma)B(\sigma)u(\sigma)\,d\sigma}_{x_\sigma(t)}}$$

$$\boxed{y(t) = \underbrace{C(t)\phi(t,t_0)x_0}_{y_n(t)} + \underbrace{\int_{t_0}^{t} \left[C(t)\phi(t,\sigma)B(\sigma) + \delta_{t-\sigma} D(\sigma)\right]u(\sigma)\,d\sigma}_{y_\sigma(t)}}$$

$$\boxed{y_\sigma(t) = \int_{t_0}^{t} \mathcal{H}(t,\sigma)u(\sigma)\,d\sigma}$$

SISTEMA DINÁMICO LINEAL de PARÁMETROS CONCENTRADOS INVARIANTE en el TIEMPO

Solución general

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

con $\begin{cases} u(t) \in \mathbb{R}^r \\ x(t) \in \mathbb{R}^n \quad \forall t \geq t_0 \\ y(t) \in \mathbb{R}^m \end{cases}$

$\left.\begin{matrix} u(t) \ \forall t \geq t_0 \\ x(t_0) = x_0 \end{matrix}\right\}$ conocidas

A, B, C, D constantes

$$\begin{cases} x(t) = \phi(t,t_0)\,x_0 + \int_{t_0}^{t} \phi(t,\sigma) B\, u(\sigma)\, d\sigma \\ y(t) = C\,\phi(t,t_0)\,x_0 + \int_{t_0}^{t} \left[C\,\phi(t,\sigma) B + \delta_{t-\sigma} D \right] u(\sigma)\, d\sigma \end{cases}$$

$$\dot{x} = A\,x$$

MATRIZ EXPONENCIAL

exponencial compleja: $e^{\lambda t}, \begin{cases} \lambda \in \mathbb{C} \\ t \in \mathbb{R} \end{cases}$

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \dots + \frac{\lambda^k t^k}{k!} + \dots = \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!}$$

definimos la matriz exp. compleja: $e^{At}, \begin{cases} A \in \mathbb{C}^{n\times n} \\ t \in \mathbb{R} \end{cases}$

$$\underline{e^{At}} \triangleq I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots + \frac{A^k t^k}{k!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Proposición:

1) e^{At} es absolutamente convergente $\forall t$ finito
2) e^{At} es uniformemente convergente en todo intervalo finito de t

Propiedades:

1) e^{At} es no-singular $\forall t$ finito.
2) $e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

PROPOSICIÓN:

$$\boxed{\phi(t, t_0) = e^{A(t-t_0)}}$$

$$\underline{x_h(t)} = e^{A(t-t_0)} x_0$$