Curso

# SISTEMAS Y CONTROL

# Clase 10

## Fotogramas de los pizarrones de clases filmadas

Prof. Rafael Canetti

Instituto de Ingeniería Eléctrica,
Facultad de Ingeniería, Universidad de la República
Montevideo, Uruguay.
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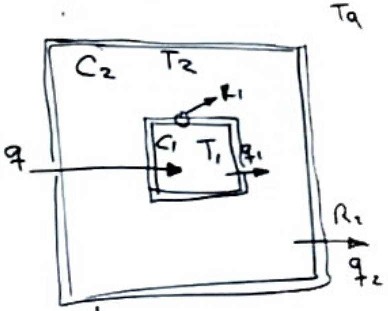
# Sistema Térmico - Ejemplo

Ejemplo:

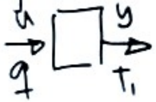



$$\begin{cases} q_1 = \dfrac{1}{R_1}(T_1 - T_2) \\ q_2 = \dfrac{1}{R_2}(T_2 - T_a) \end{cases}$$

$$\begin{cases} C_1 \dfrac{dT_1}{dt} = q - q_1 \\ C_2 \dfrac{dT_2}{dt} = q_1 - q_2 \end{cases}$$

$u = [q, T_a]'$

$y = T_1$



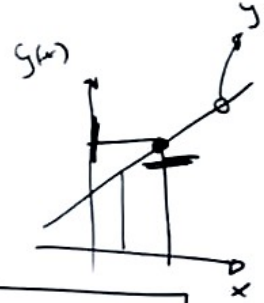

$$\begin{cases} \dfrac{dT_1}{dt} = \dfrac{1}{C_1} q - \dfrac{1}{C_1 R_1} T_1 + \dfrac{1}{C_1 R_1} T_2 \\ \dfrac{dT_2}{dt} = \dfrac{1}{C_2 R_1} T_1 - \dfrac{1}{C_2 R_1} T_2 - \dfrac{1}{C_2 R_2} T_2 + \dfrac{1}{C_2 R_2} T_a \end{cases}$$

$$x(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{T}_1(t) \\ \dot{T}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\dfrac{1}{R_1 C_1} & \dfrac{1}{R_1 C_1} \\ \dfrac{1}{R_1 C_2} & -\dfrac{1}{C_2}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \dfrac{1}{C_1} & 0 \\ 0 & \dfrac{1}{C_2 R_2} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} q(t) \\ T_a(t) \end{bmatrix}}_{u}$$

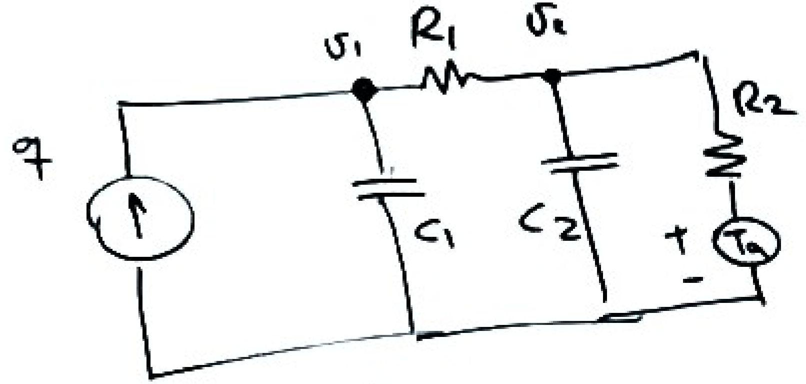
$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} q(t) \\ T_a(t) \end{bmatrix}}_{u}$$

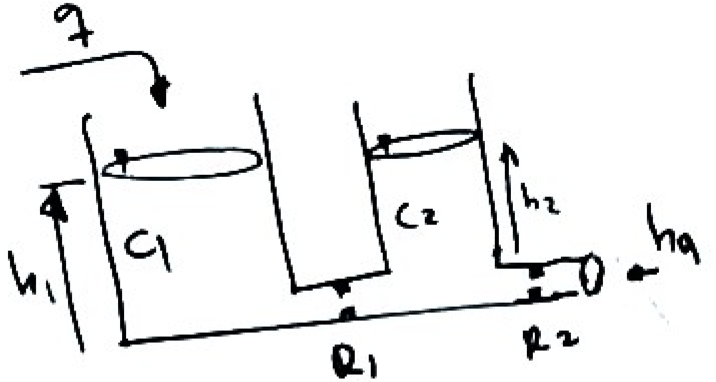
$y = ax + b$



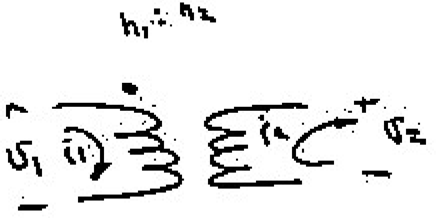

$$\boxed{\begin{aligned} \dot{X} &= Ax + Bu \\ Y &= Cx + Du \end{aligned}}$$

# Ejemplos análogos



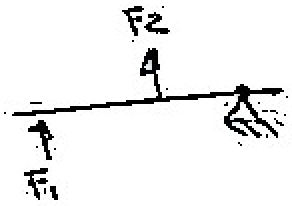



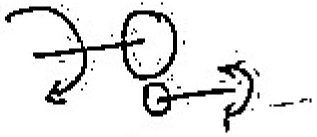
# Otros Elementos



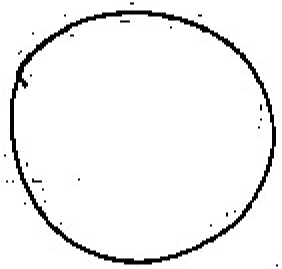

$$\frac{v_2}{v_1} = \frac{n_2}{n_1}$$

$$\frac{i_2}{i_1} = \frac{n_1}{n_2}$$



ec. gral. gases



$$P.V = NRT$$

$$\underbrace{GRT}$$

# SOLUCIÓN GENERAL del SISTEMA DINÁMICO, LINEAL de PARÁMETROS CONCENTRADOS

Sea la rep. $\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases}$ donde $\begin{cases} u(t) \in \mathbb{R}^r \\ y(t) \in \mathbb{R}^m \\ x(t) \in \mathbb{R}^n \end{cases} \forall\, t \geq t_0$

$\left.\begin{array}{l} x(t_0) = x_0 \in \mathbb{R}^n \\ u(t) \;\; \forall\, t \geq t_0 \end{array}\right\}$ conocidas

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¿$x(t)$, $y(t)$?

$$x(t) = \varphi(t, u, x_0, t_0) = \underbrace{\varphi(t, \underline{0}, x_0, t_0)}_{x_h(t)} + \underbrace{\varphi(t, u, \underline{0}, t_0)}_{x_a(t)}$$

PREVIO

# TEOREMA de PICARD

Sea la ec. dif.: (I) $\boxed{\begin{array}{l}\dot{\varphi}(t) = p(\varphi(t), t) \\ \text{con } \varphi(t_0) = \varphi_0 \text{ conocida}\end{array}}$ donde $\varphi(t) \in \mathbb{R}^n$ $\forall t \geq t_0$

¿$\varphi(t)$?

$p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

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Hip Si
① $\forall \varphi \in \mathbb{R}^n$ $p(\varphi, t)$ es secc. continua en $t$
② $\forall t \in \mathbb{R}, t \geq t_0$ $p(\varphi, t)$ es Lipschitz en $\varphi$: $\exists K(t)$ secc. continua, def. positiva /
$\forall \varphi_1, \varphi_2 \in \mathbb{R}^n : \| p(\varphi_1, t) - p(\varphi_2, t) \| \leq K(t) \| \varphi_1 - \varphi_2 \|$
$t \in \mathbb{R}, t \geq t_0$

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Tesis $\exists$
1) $\varphi(t)$ que satisface (I)
2) $\varphi(t)$ es única por $\varphi_0$
3) $\varphi(t)$ es continua

PREVIO TEOREMA de PICARD

Sea la ec. dif.: (I) $\boxed{\dot{\varphi}(t) = p(\varphi(t), t) \quad \text{con } \varphi(t_0) = \varphi_0 \text{ conocida}}$ donde $\varphi(t) \in \mathbb{R}^n$ $\forall t \geq t_0$

¿$\varphi(t)$?

$p: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

Hip si
① $\forall \varphi \in \mathbb{R}^n$ $p(\varphi, t)$ es secc. continua en $t$
② $\forall t \in \mathbb{R}, t \geq t_0$ $p(\varphi, t)$ es Lipschitz en $\varphi$: $\exists K(t)$ secc. continua, def. positiva / $\forall \varphi_1, \varphi_2 \in \mathbb{R}^n$: $\|p(\varphi_1, t) - p(\varphi_2, t)\| \leq K(t) \|\varphi_1 - \varphi_2\|$
$t \in \mathbb{R}, t \geq t_0$

Tesis $\exists$
1) $\varphi(t)$ que satisface (I)
2) $\varphi(t)$ es única por $\varphi_0$
3) $\varphi(t)$ es continua

nuestro problema ¿tiene solución?

$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$ $\qquad \dot{\varphi}(t) = p(\varphi(t), t)$

① $p(\varphi, t) = A(t)\varphi + B(t)\, u(t) \longrightarrow$ es secc. cont. en $t$

② fijamos $t$: $\varphi_1, \varphi_2$

$\|p(\varphi_1, t) - p(\varphi_2, t)\| = \|A(t)\varphi_1 - A(t)\varphi_2\| = \|A(t)(\varphi_1 - \varphi_2)\| \leq \underbrace{\|A(t)\|}_{K(t)} \|\varphi_1 - \varphi_2\|$

$\|A(t)\| \geq 0$

$\therefore$ $\exists$ solución $x(t)$, única por $x_0$
$x(t)$ es continua

## SOLUCION GENERAL del SISTEMA DINÁMICO, LINEAL de PARÁMETROS CONCENTRADOS

Sea la rep. $\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases}$ donde $\begin{cases} u(t) \in \mathbb{R}^r \\ y(t) \in \mathbb{R}^m \\ x(t) \in \mathbb{R}^n \end{cases} \forall\, t \geq t_0$

$\left.\begin{array}{l} x(t_0) = x_0 \in \mathbb{R}^n \\ u(t) \quad \forall t \geq t_0 \end{array}\right\}$ conocidas

¿$x(t), y(t)$?

$$x(t) = \varphi(t, u, x_0, t_0) = \underbrace{\varphi(t, \underline{0}, x_0, t_0)}_{x_h(t)} + \underbrace{\varphi(t, u, \underline{0}, t_0)}_{x_\sigma(t)}$$

Hip $A(t), B(t), C(t), D(t)$ $u(t)$ son secc. continuas en $t$

① Cálculo de $x_h(t) = \varphi(t, \underline{0}, x_0, t_0)$

es la sol. de $\boxed{\begin{array}{l}\dot{x}_h(t) = A(t)\,x_h(t) \\ \text{con } x(t_0) = x_0\end{array}}$

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix} = \sum_{i=1}^{n} x_{0i}\, e_i \quad \text{con } e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{i-ésima fila}$$

$$x_h(t) = \varphi(t, \underline{0}, x_0, t_0) = \varphi\left(t, \underline{0}, \sum_{i=1}^{n} x_{0i}\, e_i, t_0\right) = \sum_{i=1}^{n} \varphi(t, \underline{0}, x_{0i}\, e_i, t_0) =$$

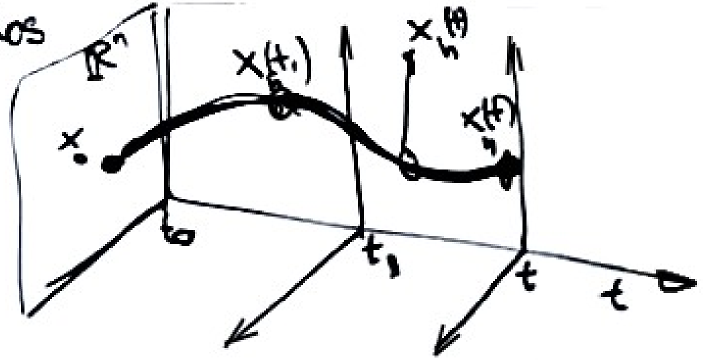
$$x_h(t) = \sum_{i=1}^{n} x_{0i}\, \varphi(t, \underline{0}, e_i, t_0)$$

$$\Rightarrow x_h(t) = \begin{bmatrix} \varphi(t, \underline{0}, e_1, t_0) & \varphi(t, \underline{0}, e_2, t_0) & \dots & \varphi(t, \underline{0}, e_n, t_0) \end{bmatrix} \underbrace{\begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix}}_{x_0}$$

# Matriz de Transición de Estados

$$\boxed{x_h(t) = \Phi(t, t_0)\, x_0}$$

$\Phi$ : matriz de transición de estados




$$\boxed{\phi(t, t_0) = \phi(t, t_1)\, \phi(t_1, t_0)}$$ ⟵ propiedad

$t \geq t_1 \geq t_0$

$$x_h(t) = \underbrace{\phi(t, t_0)}\, x_0$$

$$x_h(t) = \phi(t, t_1)\, x_h(t_1) = \underbrace{\phi(t, t_1) \cdot \phi(t_1, t_0)\, x_0} \quad \forall x_0 \in \mathbb{R}^n \Longrightarrow$$

# Solución General

Proposición

$$\boxed{x(t) = \underbrace{\Phi(t,t_0)\, x_0}_{x_h(t)} + \underbrace{\int_{t_0}^{t} \Phi(t,\sigma)\, B(\sigma)\, \underline{u}(\sigma)\, d\sigma}_{x_\sigma(t)}}$$