1) Representación en variables de estado:
$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} kCB - \dfrac{Q_i B}{V} \\ -\alpha kCB - \dfrac{Q_i C}{V} + \dfrac{C_i Q_i}{V} \end{bmatrix} \\ B = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} B \\ C \end{bmatrix} \end{cases}.$$

2) Punto de operación en régimen: $\begin{bmatrix} B^0 \\ C^0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\alpha}\left(C_i^0 - \dfrac{Q_i^0}{kV}\right) \\ \dfrac{Q_i^0}{kV} \end{bmatrix}$.

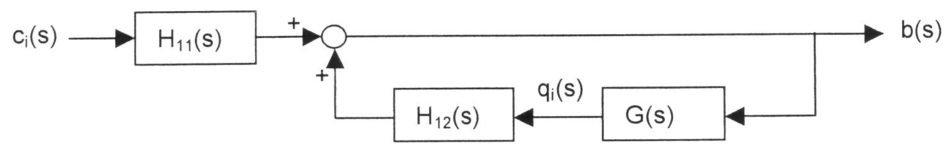
Condición de existencia del punto de operación: $C_i^0 > \dfrac{Q_i^0}{kV}$ (para que $B^0 > 0$). De ahora en adelante se asume que se verifica esta condición.

3) Linealización:
$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} b \\ c \end{bmatrix} = A\begin{bmatrix} b \\ c \end{bmatrix} + B\begin{bmatrix} c_i \\ q_i \end{bmatrix} \\ b = C\begin{bmatrix} b \\ c \end{bmatrix} \end{cases} ;\ A = \begin{bmatrix} 0 & \dfrac{kVC_i^0 - Q_i^0}{\alpha V} \\ \dfrac{-\alpha Q_i^0}{V} & -kC_i^0 \end{bmatrix} ;\ B = \begin{bmatrix} 0 & -\dfrac{kVC_i^0 - Q_i^0}{\alpha kV^2} \\ \dfrac{Q_i^0}{V} & \dfrac{kVC_i^0 - Q_i^0}{kV^2} \end{bmatrix} ;\ C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

$$C^0 \approx 1{,}594\,\frac{g}{m^3} ;\ B^0 \approx 399{,}2\,\frac{g}{m^3} ;\ A = \begin{bmatrix} 0 & 0{,}2505\,s^{-1} \\ -0{,}002\,s^{-1} & -0{,}502\,s^{-1} \end{bmatrix} ;\ B \approx \begin{bmatrix} 0 & -399{,}2\,\dfrac{g}{m^6} \\ 0{,}001 s^{-1} & 798{,}4\,\dfrac{g}{m^6} \end{bmatrix}.$$

Matriz de transferencia:
$$H(s) = C(sI - A)^{-1}B = \begin{bmatrix} H_{11}(s) & H_{12}(s) \end{bmatrix} = \begin{bmatrix} \dfrac{K_1}{(s+\omega_1)(s+\omega_2)} & \dfrac{-K_2}{s+\omega_2} \end{bmatrix} ;$$
$$K_1 = \frac{Q_i^0\left(kVC_i^0 - Q_i^0\right)}{\alpha V^2} > 0 ;\ K_2 = \frac{kVC_i^0 - Q_i^0}{\alpha kV^2} > 0 ;\ \omega_1 = \frac{Q_i^0}{V} ;\ \omega_2 = \frac{kVC_i^0 - Q_i^0}{V} > 0 .$$
$$K_1 = 2{,}505 x 10^{-4}\,s^{-2} ;\ K_2 \approx 399{,}2\,\frac{g}{m^6} ;\ \omega_1 = 1{,}0 x 10^{-3}\,s^{-1} ;\ \omega_2 = 0{,}501\,s^{-1} .$$

4) Control en tiempo continuo:



Función de transferencia en lazo cerrado: $H_{LC}(s) = \dfrac{b(s)}{c_i(s)} = \dfrac{K_1}{(s+\omega_1)(s+\omega_2+K_2G(s))}$.

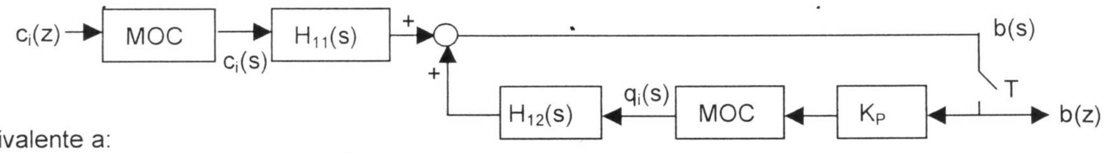
Si $c_i(t) = E \cdot 1(t)$ (escalón de amplitud E), entonces $c_i(s) = \dfrac{E}{s}$ y se tiene:
$$\lim_{s\to 0} s(b(s)) = \lim_{s\to 0} s\left(H_{LC}(s)\frac{E}{s}\right) = \lim_{s\to 0} s\frac{K_1}{(s+\omega_1)(s+\omega_2+K_2G(s))}\frac{E}{s} = \lim_{s\to 0} \frac{K_1E}{\omega_1(\omega_2+K_2G(s))} .$$
Para que este límite sea 0, se requiere que $\lim_{s\to 0} G(s) = \infty$.

Tomando $G(s) = \dfrac{K_I}{s}$ con $K_I > 0$, resulta la función de transferencia en lazo cerrado estable:
$$H_{LC}(s) = \frac{b(s)}{c_i(s)} = \frac{K_1 s}{(s+\omega_1)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} ;\ \zeta = \frac{\omega_2}{2\sqrt{K_2K_I}} ;\ \omega_n = \sqrt{K_2K_I} .$$

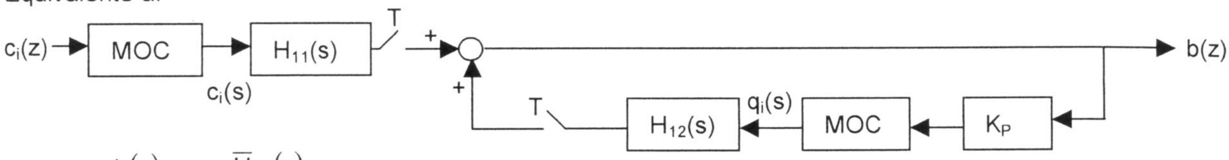
Como $H_{LC}(s)$ es estable, $0 = \lim\limits_{s\to 0} s\left(H_{LC}(s)\dfrac{E}{s}\right) = \lim\limits_{t\to+\infty} b(t)$.

Para obtener respuesta críticamente amortiguada: $K_I = \dfrac{\omega_2{}^2}{4K_2} = \dfrac{\alpha k}{4}\left(kVC_i^0 - Q_i^0\right) = 1{,}572\times10^{-4}\ \dfrac{m^6}{g\cdot s^2}$.

5) Control en tiempo discreto:



Equivalente a:



$$T_{LC}(z) = \frac{b(z)}{c_i(z)} = \frac{\overline{H}_{11}(z)}{1-\overline{H}_{12}(z)K_P} \text{ donde:}$$

$$\overline{H}_{11}(z) = \frac{z-1}{z}Z\left\{L^{-1}\left\{\frac{H_{11}(s)}{s}\right\}\Bigg|_{t=kT}\right\} = \frac{K_1}{\omega_1\omega_2}\left[1+\frac{z-1}{\omega_1-\omega_2}\left(\frac{\omega_2}{z-e^{-\omega_1 T}} - \frac{\omega_1}{z-e^{-\omega_2 T}}\right)\right] = \left(\frac{K_1}{\omega_1\omega_2}\right)\frac{b_1 z + b_2}{(z-a)(z-b)},$$

$$\overline{H}_{12}(z) = \frac{z-1}{z}Z\left\{L^{-1}\left\{\frac{H_{12}(s)}{s}\right\}\Bigg|_{t=kT}\right\} = -\left(\frac{K_2}{\omega_2}\right)\frac{1-e^{-\omega_2 T}}{z-e^{-\omega_2 T}} = -\left(\frac{K_2}{\omega_2}\right)\frac{1-b}{z-b};$$

siendo $a = e^{-\omega_1 T}$; $b = e^{-\omega_2 T}$ y $b_1 = \dfrac{\omega_2(1-a)-\omega_1(1-b)}{\omega_2-\omega_1}$; $b_2 = \dfrac{\omega_1(1-b)a - \omega_2(1-a)b}{\omega_2-\omega_1}$.

Entonces: $T_{LC}(z) = \left(\dfrac{K_1}{\omega_1\omega_2}\right)\dfrac{b_1 z + b_2}{(z-a)\left(z-\left(b-\dfrac{K_2}{\omega_2}(1-b)K_P\right)\right)}$.

El sistema es estable si y sólo si: $\left|b-\dfrac{K_2}{\omega_2}(1-b)K_P\right| < 1$.

$$\left|b-\frac{K_2}{\omega_2}(1-b)K_P\right| < 1 \Leftrightarrow -1 < b-\frac{K_2}{\omega_2}(1-b)K_P < 1 \Leftrightarrow$$

$$\Leftrightarrow -\left(\frac{\omega_2}{K_2}\right) < K_P < \frac{1+b}{1-b}\left(\frac{\omega_2}{K_2}\right) \Leftrightarrow -1{,}255\times10^{-3}\ \frac{m^6}{g\cdot s} \approx -(\alpha kV) < K_P < \frac{1+e^{-\omega_2 T}}{1-e^{-\omega_2 T}}(\alpha kV) \approx 5{,}114\times10^{-3}\ \frac{m^6}{g\cdot s}$$

Si el sistema es estable; ante un escalón unitario en $c_i(k)$ la salida $b(k)$ tiende asintóticamente al valor:

$$\lim_{k\to+\infty} b(k) = \lim_{z\to1}(z-1)\left(T_{LC}(z)\frac{z}{z-1}\right) = \frac{\dfrac{K_1}{\omega_1\omega_2}}{1+\dfrac{K_2}{\omega_2}K_P} = \frac{1}{\alpha+\dfrac{K_P}{kV}} = \frac{1}{2+\dfrac{K_P}{6{,}275\times10^{-4}\ \dfrac{m^6}{g\cdot s}}}.$$