Parcial 1 - 12 de mayo de 2007
SOLUCIONES

1. (a)

$$\mathbf{p} = \frac{10^3}{10^5} = \frac{\mathbf{1}}{\mathbf{100}}$$

(b)

$$\begin{aligned} C &= \text{capicúa} \qquad I = 3 \text{ dígitos iguales} \\ P(C/I) &= \frac{P(C \cap I)}{P(I)} \\ P(C \cap I) &= \frac{2 \times 10 \times 9}{10^5} \\ P(I) &= \frac{C_3^5 \times 10 \times 9 \times 9}{10^5} \\ P(C/I) &= \frac{2 \times 10 \times 9}{C_3^5 \times 10 \times 9 \times 9} = \frac{\mathbf{1}}{\mathbf{45}} \end{aligned}$$

(c) Sea X el número de boletos capicúas en los siguientes 7 viajes

$$\begin{aligned} X &\sim Bin(7, p) \\ \mathbf{P}(\mathbf{X} \geq \mathbf{2}) &= 1 - P(X = 0) - P(X = 1) \\ &= \mathbf{1} - \left(\frac{\mathbf{99}}{\mathbf{100}}\right)^{\mathbf{7}} - \frac{\mathbf{7}}{\mathbf{100}}\left(\frac{\mathbf{99}}{\mathbf{100}}\right)^{\mathbf{6}} \end{aligned}$$

(d) Sea X el número de boletos capicúas en los siguientes n viajes

$$\begin{aligned} X &\sim Bin(n, p) \\ P(X \geq 1) &> \frac{1}{2} \\ 1 - (1-p)^n &> \frac{1}{2} \\ (1-p)^n &< \frac{1}{2} \\ n &> \frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{99}{100}\right)} \approx 68.9 \Rightarrow \mathbf{n} = \mathbf{70} \end{aligned}$$

(e) Sea X el número de viajes necesarios hasta obtener 5 boletos capicúas.

$$\begin{aligned} X &\sim BN(5, p) \\ E(X) &= \frac{5}{p} = \frac{5}{\frac{1}{100}} = \mathbf{500} \end{aligned}$$

2. (a) i.

$$a\int_{-\infty}^{0}\frac{1}{1+x^2}dx = \frac{a}{2}\pi=\frac{1}{2}\Rightarrow \mathbf{a}=\frac{\mathbf{1}}{\boldsymbol{\pi}}$$
$$\int_{0}^{\infty}be^{-x}dx = b\int_{0}^{\infty}e^{-x}dx=\mathbf{b}=\frac{\mathbf{1}}{\mathbf{2}}$$

ii. $$F_Z(z)=\begin{cases}\frac{1}{\pi}\int_{-\infty}^{z}\frac{1}{1+t^2}dt=\frac{\mathbf{1}}{\boldsymbol{\pi}}\left(\arctan\mathbf{z}+\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\pi}\right) & \text{si } z<0\\ \frac{1}{\pi}\int_{-\infty}^{0}\frac{1}{1+t^2}dt+\frac{1}{2}\int_{0}^{z}e^{-t}dt=\mathbf{1}-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e^{-z}} & \text{si } z\geq 0\end{cases}$$

iii.

$$\frac{1}{\pi}\int_{-\infty}^{0}\frac{t}{1+t^2}dt \sim \int_{-\infty}^{0}\frac{1}{t}dt=-\infty$$
$$\frac{1}{2}\int_{0}^{\infty}te^{-t}dt = \frac{1}{2}\Rightarrow$$
$$E(X) = -\infty$$

(b) i.

$$f_X(x)=\int_{-\infty}^{\infty}f_{XY}(x,y)\,dy=\begin{cases}\frac{1}{\pi}\frac{1}{1+x^2}\int_{0}^{1}dy & \text{si } x<0\\ \frac{1}{2}e^{-x}\int_{0}^{1}dy & \text{si } x\geq 0\end{cases}=\begin{cases}\frac{\mathbf{1}}{\boldsymbol{\pi}}\frac{\mathbf{1}}{\mathbf{1}+\mathbf{x}^2} & \text{si } x<0\\ \frac{\mathbf{1}}{\mathbf{2}}\mathbf{e}^{-x} & \text{si } x\geq 0\end{cases}$$

ii.

$$f_Y(y) = \int_{-\infty}^{\infty}f_{XY}(x,y)\,dx=\begin{cases}\frac{1}{\pi}\int_{-\infty}^{0}\frac{1}{1+x^2}dx+\frac{1}{2}\int_{0}^{+\infty}e^{-x}dx\\ \int_{-\infty}^{\infty}0\,dx\end{cases}=\begin{cases}\mathbf{1} & \text{si } y\in[0,1]\\ \mathbf{0} & \text{si } y\notin[0,1]\end{cases}$$
$$\Rightarrow\quad Y\sim U[0,1]$$

iii.

$$f_X(x)\times f_Y(y)=f_{XY}(x,y)\ \forall\ (x,y)\in\mathbb{R}^2\Rightarrow \textbf{X e Y son independientes}$$

3. (a) i.

$$\begin{aligned} F_Z(z) &= P(Z \le z) = 1 - P(Z \ge z) = 1 - P(X \ge z)\, P(Y \ge z) \\ &= 1 - (1 - F_X(z))(1 - F_Y(z)) \\ &= \begin{cases} 1 - \left(e^{-\mu z}\right)\left(e^{-\lambda z}\right) & \text{si} \quad z \ge 0 \\ 0 & \text{si} \quad z < 0 \end{cases} = \begin{cases} 1 - e^{-(\mu+\lambda) z} & \text{si} \quad z \ge 0 \\ 0 & \text{si} \quad z < 0 \end{cases} \\ &\Rightarrow \mathbf{Z} \sim \exp(\boldsymbol{\mu} + \boldsymbol{\lambda}) \end{aligned}$$

ii.

$$\begin{aligned} f_{XY}(x,y) &= \begin{cases} \mu\lambda e^{-\mu x} e^{-\lambda y} \text{ si } x \ge 0 \text{ y } y \ge 0 \\ 0 \text{ en otro caso} \end{cases} \\ A &= \{(x,y) : x \ge 0 \text{ y } y \ge 0\} \\ B &= \{(x,y) : x \le y\} \\ \mathbf{P}(\mathbf{X} \le \mathbf{Y}) &= \iint_B f_{XY}(x,y)\, dxdy = \iint_{B \cap A} \mu\lambda e^{-\mu x} e^{-\lambda y} dxdy + \iint_{B \cap A^C} 0 dxdy \\ &= \int_0^\infty \left[\int_x^\infty \mu\lambda e^{-\mu x} e^{-\lambda y} dy \right] dx = \int_0^\infty \mu e^{-\mu x} \left[\int_x^\infty \lambda e^{-\lambda y} dy \right] dx \\ &= \int_0^\infty \mu e^{-\mu x} e^{-\lambda x} dx = \int_0^\infty \mu e^{-(\mu+\lambda) x} dx = \frac{\boldsymbol{\mu}}{\boldsymbol{\mu} + \boldsymbol{\lambda}} \end{aligned}$$

(b)

$$\begin{aligned} P(T_1 = T_2) &= P(X_3 \le \min\{X_1, X_2\}) + \underbrace{P(X_1 = X_2, X_1 < X_3)}_{=0} \\ \min\{X_1, X_2\} &\sim \exp(\lambda_1 + \lambda_2) \text{ y } X_3 \sim \exp(\lambda_3) \text{ e independientes} \\ \overset{\textbf{a.ii}}{\Rightarrow} P(X_3 \le \min\{X_1, X_2\}) &= \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \\ &\Rightarrow P(T_1 = T_2) = \frac{\boldsymbol{\lambda}_3}{\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3} \end{aligned}$$