9. V.A. DISCRETAS III

1

Ej 1

x	-2	-1	0	1	2	suma
$p(x)$	$1/15$	$2/15$	$3/15$	$4/15$	$5/15$	1
$x\,p(x)$	$-2/15$	$-2/15$	0	$4/15$	$10/15$	$10/15$
$(x-\frac{2}{3})^2$	$64/9$	$25/9$	$4/9$	$1/9$	$16/9$	$110/9$

$E(X) = 10/15 = 2/3 \qquad Var(X) = 110/9$

Ej 2

1.

x	-1	0	1	
$p(x)$	$1/8$	$2/8$	$5/8$	
$x\,p(x)$	$-1/8$	0	$5/8$	$1/2 = E(X)$.

2.

x	-1	0	1
$y = x^2$	1	0	1
	$1/8$	$2/8$	$5/8$

$\Rightarrow$

y	0	1
$p(y)$	$2/8$	$6/8$

$\Rightarrow E(Y) = 6/8$.

3.

x	-1	0	1	
x^2	1	0	1	
$x^2 p(x)$	$1/8$	0	$5/8$	$6/8 = E(X^2)$.

4. $Var(X) = E(X^2) - E(X) = 6/8 - 1/4 = 6/8 - 2/8 = 4/8 = 1/2$

2

Ej 3 $E(X) = 5$ y $Var(X) = 2$

$Var(X) = E(X^2) - E(X)^2$

$2 = E(X^2) - 25 \Rightarrow E(X^2) = 27$.

Ej 4

x	0	2	3	suma
$p(x)$	$3/10$	$1/10$	$6/10$	1
$x\,p(x)$	0	$2/10$	$18/10$	$2 = E(X)$
$x^2 p(x)$	0	$4/10$	$54/10$	$58/10 = E(X^2)$

$Var(X) = \frac{58}{10} - \frac{40}{10} = 18/10$.

y	3	3	12
prob.	$3/10$	$1/10$	$6/10$

y	3	12	suma
$p(y)$	$4/10$	$6/10$	1
$y\,p(y)$	$12/10$	$72/10$	$84/10 = E(Y)$
fda $F(y)$	$4/10$	1	

$F_Y(7) = 4/10$.

Ej 5 1. $S_n = X_1 + \dots + X_n \qquad E(S_n) = \frac{7}{2}\cdot n \quad Var(S_n) = \frac{35}{12}\cdot n$

2. $\overline{X}_n = \frac{S_n}{n} \qquad E(\overline{X}_n) = 7/2 \quad Var(\overline{X}_n) = \frac{35}{12\,n} \xrightarrow[n\to\infty]{} 0$.

3

Ej 6

1. $E(e^{-X}) = \sum_{k=1}^{\infty} e^{-k} p(1-p)^{k-1} = ep \sum_{k=1}^{\infty} \left(\frac{(1-p)}{e} \right)^{k-1}$

$= ep \sum_{j=0}^{\infty} \left(\frac{1-p}{e} \right)^{j} = ep \frac{1}{1 - \frac{(1-p)}{e}} = e^2 p \big/ e-(1-p)$.

2. $E\left(\frac{1}{X}\right) = \sum_{k=1}^{\infty} \frac{1}{k} p(1-p)^{k-1} = p \sum_{k=1}^{\infty} \frac{(1-p)^{k-1}}{k}$

$= \frac{p}{(1-p)} \sum_{k=1}^{\infty} \frac{(1-p)^k}{k} = \frac{-p}{1-p} \sum_{k=1}^{\infty} \int (1-p)^{k-1} dp$

$\frac{(1-p)^k}{k} = -\int (1-p)^{k-1} dp$

$= \frac{-p}{1-p} \int \underbrace{\sum_{k=1}^{\infty} (1-p)^{k-1}}_{\frac{1}{1-(1-p)} = \frac{1}{p}} dp = \frac{-p}{1-p} \int \frac{1}{p} dp$

$= \frac{-p}{1-p} \ln(p)$

$\Rightarrow \boxed{E\left(\frac{1}{X}\right) = \frac{p}{1-p} \ln(1/p)}$

Ej 7

1. $Var(X+Y) = var(X) + var(Y) + 2cov(X,Y)$

$Var(X-Y) = var(X) + var(Y) - 2cov(X,Y)$

$var(X+Y) + var(X-Y) = 2var(X) + 2var(Y)$

4

2. $Cov(X+Y, X-Y) = E((X+Y)(X-Y)) - E(X+Y)E(X-Y)$

$= E(X^2-Y^2) - (E(X)^2 - E(Y)^2)$

$= E(X^2) - E(X)^2 - (E(Y^2) - E(Y)^2)$

$= var(X) - var(Y) = 0$.

Ej 8

$E(X) = 0 \qquad Y = e^{|X|}$

$E(XY) = \sum_{k=-a}^{a} k e^k = -ae^a - \dots - 1 \cdot e^1 + 0 \cdot e^0 + 1e^1 + \dots + ae^a = 0$

$\Rightarrow cov(X,Y) = E(XY) - E(X)E(Y) = 0 - 0 = 0$.

X e Y no son independientes.

Ej 9

$X \sim Ber(p) \qquad XY = \begin{cases} 1 & \text{si } X=1 \text{ e } Y=1 \\ 0 & \text{si no} \end{cases}$

$Y \sim Ber(q)$

$E(XY) = P(X=1, Y=1)$.

Si $cov(X,Y) = 0 \Rightarrow E(XY) = E(X)E(Y)$

$\Rightarrow P(X=1, Y=1) = P(X=1)P(Y=1)$.

Tomando complementos se ve que X e Y son independientes.
Por ejemplo:

$P(X=0, Y=1) + P(X=1, Y=1) = P(Y=1)$

$P(X=0, Y=1) = P(Y=1) - P(X=1, Y=1) = P(Y=1) - P(X=1)P(Y=1)$

$= (1 - P(X=1))P(Y=1) = P(X=0)P(Y=1)$

El resto es igual.

Ej 10 $\quad n$ lantamientos

(5)

$$X_i = \begin{cases} 1 & \text{si sale 1 en el i-ésimo} \\ 0 & \text{si no} \end{cases}$$

$$Y_i = \begin{cases} 1 & \text{si sale 6 en el i-ésimo} \\ 0 & \text{si no} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i = \text{nº de 1's} \qquad E(X_i) = 1/6 \quad Var(X_i) = \frac{5}{36}$$

$$Y = \sum_{i=1}^{n} Y_i = \text{nº de 6's} \qquad E(Y_i) = 1/6 \quad Var(Y_i) = \frac{5}{36}$$

$$E(X) = \sum_{i=1}^{n} E(X_i) = n/6 \ , \quad E(Y) = n/6$$

$$Var(X) = \frac{5n}{36} \qquad Var(Y) = \frac{5n}{36}$$

$$E(XY) = E\left(\sum_{i,j=1}^{n} X_i Y_j\right) = E\left(\sum_{i \neq j} X_i Y_j\right)$$

$$= \sum_{i \neq j} E(X_i)E(Y_j) = \frac{n(n-1)}{36}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{n(n-1)}{36} - \frac{n^2}{36} = \frac{-n}{36}.$$

10. V.A. CONTINUAS I

Ej 1 (V), (vii) y (viii) si.

Ej 2

1. $f(x) = \frac{c}{x^4}$ $\quad x > 1$ $\qquad \int_1^\infty \frac{c}{x^4}\,dx = c\,\frac{x^{-3}}{-3}\Big|_1^\infty = \frac{c}{3} = 1$

$\Rightarrow c = 3.$

$$F(x) = \int_{-\infty}^x f(y)\,dy = \begin{cases} 0 & \text{si } x \leq 1 \\ 1 - \frac{1}{x^3} & \text{si } x > 1 \end{cases}$$

$$E(X) = 3\int_1^{+\infty} \frac{1}{x^3}\,dx = 3\,\frac{x^{-2}}{-2}\Big|_1^\infty = 3/2$$

$$E(X^2) = 3\int_1^{+\infty} \frac{1}{x^2}\,dx = 3\,\frac{x^{-1}}{-1}\Big|_1^\infty = 3$$

$$Var(X) = 3 - 9/4 = \frac{12-9}{4} = 3/4.$$

2. $f(x) = c\,x(1-x)$ $\quad 0 < x < 1$

$$c\int_0^1 x - x^2\,dx = c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = c\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{c}{6} \Rightarrow \boxed{c = 6}$$

$$F(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right] & \text{si } 0 < x < 1 \\ 1 & \text{si } x \geq 1 \end{cases}$$

$$E(X) = 6\int_0^1 x^2 - x^3\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{6}{12} = \frac{1}{2}$$

$$E(X^2) = 6\int_0^1 x^3 - x^4\,dx = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right]$$

$$= 6/20 = 3/10$$

$$Var(X) = 3/10 - 1/4 = \frac{12-10}{40} = \frac{2}{40} = \frac{1}{20}.$$

3. $f(x) = c\,x^2(1-x)^2$ $\quad 0 < x < 1$

$= c\,x^2(1 - 2x + x^2)$

$= c\,(x^2 - 2x^3 + x^4)$

$$c\int_0^1 x^2 - 2x^3 + x^4\,dx = c\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= c\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]$$

$$= c\left[\frac{10-15+6}{30}\right] = c\,\frac{1}{30} = 1$$

$\Rightarrow \boxed{c = 30}$

$$F(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ 30\left[\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right] & \text{si } 0 \leq x \leq 1 \\ 1 & \text{si } x \geq 1. \end{cases}$$

$$E(X) = 30\int_0^1 x^3 - 2x^4 + x^5\,dx = 30\left[\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right]_0^1$$

$$= 30\left[\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right] = 30\left[\frac{30-48+20}{120}\right] = \frac{60}{120} = 1/2.$$

$$E(X^2) = 30\int_0^1 x^4 - 2x^5 + x^6\,dx = 30\left[\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right] = \frac{21-35+15}{105}\cdot 30$$

$$= \frac{30}{105} = \frac{6}{21} = \frac{2}{7}. \quad Var(X) = \frac{2}{7} - \frac{1}{4} = \frac{8-7}{28} = \frac{1}{28}.$$

Ej 3 $f(x) = x + ax^2$ en $[0,1]$.

1. $\int_0^1 x + ax^2\,dx = \left[\frac{x^2}{2} + a\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{a}{3} = 1$

$\Rightarrow \boxed{a = 3/2}$

2. $F(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{x^2 + x^3}{2} & \text{si } 0 \le x \le 1 \\ 1 & \text{si } x > 1. \end{cases}$

3. $P(1/2 < X \le 1) = F(1) - F(1/2) = 1 - \left(\frac{1/4 + 1/8}{2}\right)$

$= 1 - \frac{3}{16} = \frac{13}{16}.$

4. $E(X) = \int_0^1 x^2 + \frac{3}{2}x^3\,dx = \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^4}{4}\right]_0^1 = \frac{1}{3} + \frac{3}{8}$

$= \frac{8+9}{24} = \frac{17}{24}.$

$E(X^2) = \int_0^1 x^3 + \frac{3}{2}x^4\,dx = \left[\frac{x^4}{4} + \frac{3}{2}\cdot\frac{x^5}{5}\right]_0^1 = \frac{1}{4} + \frac{3}{10} = \frac{22}{40} = \frac{11}{20}$

$Var(X) = \frac{11}{20} - \frac{289}{576} = \frac{6336 - 5780}{11520} = \frac{556}{11520} \approx 0.048.$

Ej 4 $F(x) = \begin{cases} 0 & \text{si } x < 0 \\ x^2 & \text{si } 0 \le x \le 1 \\ 1 & \text{si } x > 1 \end{cases}$

1. $P(1/2 < X < 3/4) = F(3/4) - F(1/2) = \frac{9}{16} - \frac{1}{4} = 5/16$

2. $p(x) = F'(x) = 2x$ en $[0,1]$.

Ej 5 $F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 2x^2 - x^4 & \text{si } 0 \le x \le 1 \\ 1 & \text{si } x > 1 \end{cases}$

1. $P(1/4 \le X \le 3/4) = F(3/4) - F(1/4)$

$= 2\frac{9}{16} - \frac{81}{256} - \left(2\cdot\frac{1}{16} - \frac{1}{256}\right)$

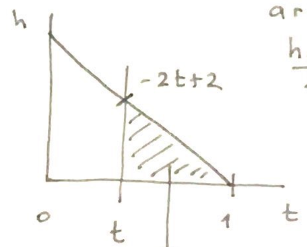
$= \frac{2\cdot 8}{16} - \frac{80}{256} = 1 - \frac{80}{256} = \frac{176}{256}$

$= \frac{11\cdot 16}{16\cdot 16} = \frac{11}{16}$

2. $p(x) = 4x - 4x^3$ en $[0,1]$.

Ej 6

1.



$\frac{2(1-t)^2}{2}$

area = 1

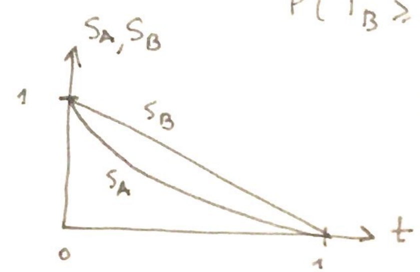
$\frac{h\cdot 1}{2} = 1 \Rightarrow h = 2$

$\begin{cases} y = ax + b, & b = 2 \\ 0 = a\cdot 1 + 2, & a = -2 \end{cases}$

$P_A(t) = -2t + 2$

$P(T_A \ge t) = (1-t)^2.$

$P(T_B \ge t) = 1 - t$



Prefiero el A, pues es más probable que termine antes.

⑤

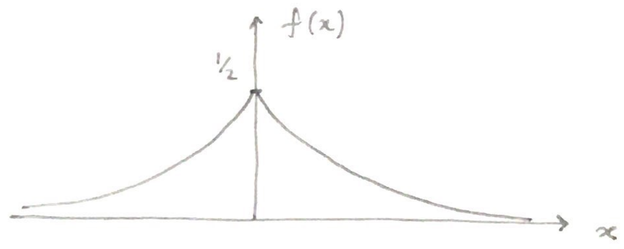
2. $E(T_O) = 1/2 \quad E(T_A) = \int_0^1 2t(1-t)\,dt = 2\int_0^1 t - t^2\,dt$

$$= 2\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{2}{6}$$

$$= 1/3.$$

Ej 7 $\quad f(x) = \dfrac{1}{2(1+|x|)^2} \quad x \in \mathbb{R}$

1.



2. $P(-1 < X < 2) = \int_{-1}^{2} f(x)\,dx$

$$= \int_{-1}^{0} \frac{1}{2(1-x)^2}\,dx + \int_0^2 \frac{1}{2(1+x)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{1}{1-x}\right]_{-1}^{0} + \frac{1}{2}\left[\frac{-1}{1+x}\right]_0^2$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right] + \frac{1}{2}\left[-\frac{1}{3} + 1\right]$$

$$= \frac{1}{4} + \frac{1}{3} = 7/12.$$

3. $P(|X| > 1) = 2\int_1^{+\infty} \frac{1}{2(1+x)^2}\,dx = \left[\frac{-1}{1+x}\right]_1^{+\infty} = 1/2$

4. No, pues $\int_{-\infty}^{+\infty} \frac{x}{2(1+|x|)^2}\,dx$ no converge.

⑥

Ej 8 $\quad X$ con densidad $\quad f(x) = \begin{cases} 0 & \text{si } x \le 0 \\ k x e^{-x/3} & \text{si } x > 0 \end{cases}$

1. $\int_0^{+\infty} k x e^{-x/3}\,dx = \begin{bmatrix}\text{integrando} \\ \text{por partes}\end{bmatrix} = 9k = 1$

$\Rightarrow k = 1/9$

2. $E(X) = \int_0^{+\infty} \frac{1}{9} x^2 e^{-x/3}\,dx = \frac{54}{9} = 6.$

3. $P(X \ge 27) = \int_{27}^{+\infty} \frac{1}{9} x e^{-x/3}\,dx = \frac{90}{9e^9} = \frac{10}{e^9} \approx 1.2 \times 10^{-3}$

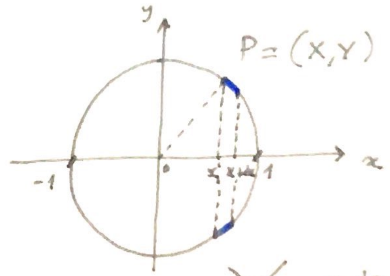
4. $Y = X_1 + \dots + X_7$ nº de cortes en 1 semana.

$P(Y = 2) = \binom{7}{2}(1.2\times10^{-3})^2(1 - 1.2\times10^{-3})^5.$

5. $P(Y \ge 1) = 1 - P(Y=0) = 1 - (1 - 1.2\times10^{-3})^7.$

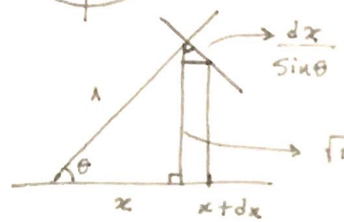
Ej 9

1.



$P(x \le X \le x+dx) = \frac{2\,dx}{\sqrt{1-x^2}} \cdot \frac{1}{2\pi}$

$\Rightarrow \boxed{p(x) = \frac{1}{\pi\sqrt{1-x^2}}}$



$\frac{dx}{\sin\theta} = \frac{dx}{\sqrt{1-x^2}}$

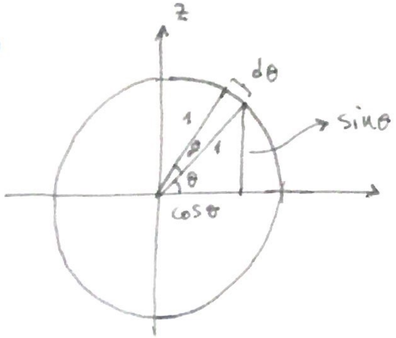
$\sqrt{1-x^2} = \sin\theta$

$-1 \le x \le 1.$

2. La densidad de $Y=|X|$ es: $p(y) = \dfrac{2}{\pi\sqrt{1-y^2}}$ $\quad 0 \le y \le 1$.

Ej 10

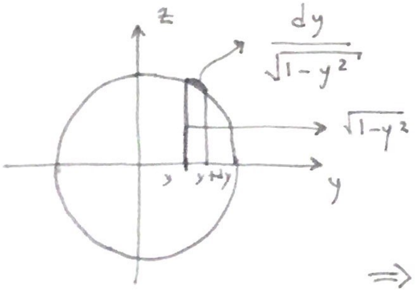
1.



$$p(\theta)d\theta = \frac{2\pi\cos(\theta)d\theta}{4\pi}$$

$$\boxed{p(\theta) = \frac{1}{2}\cos(\theta)}$$

2.



$$\text{Area} = 2\pi\sqrt{1-y^2}\cdot\frac{dy}{\sqrt{1-y^2}} = 2\pi\,dy$$

$$p(y) = \frac{2\pi}{4\pi} = 1/2 \text{ en } [-1,1].$$

$\Rightarrow$ Y es uniforme en $[-1,1]$.