3. Axiomas de Kolmogorov

Ej 1

1. Si $A \subset B$ entonces $B = A \cup B \setminus A$ y

$$P(B) = P(A) + P(B \setminus A) \Rightarrow P(B \setminus A) = P(B) - P(A)$$

Como $P(B \setminus A) \geq 0 \Rightarrow P(B) \geq P(A)$

2. Para la unión:

$$\left.\begin{array}{l} P(A \cup B) \geq P(A) \text{ ya que } A \subset A \cup B \\ P(A \cup B) \geq P(B) \text{ ya que } B \subset A \cup B \end{array}\right\} P(A \cup B) \geq \max\{P(A), P(B)\}$$

Para la intersección:

$$\left.\begin{array}{l} P(A \cap B) \leq P(A) \text{ ya que } A \cap B \subset A \\ P(A \cap B) \leq P(B) \text{ ya que } A \cap B \subset B \end{array}\right\} P(A \cap B) \leq \min\{P(A), P(B)\}$$

Ej 2 $P(A) = 1/3$, $P(B) = 1/2$

1. $A^c \cap B$. Si $A \cap B = \emptyset \Rightarrow B \subset A^c \Rightarrow A^c \cap B = B$

$$P(A^c \cap B) = P(B) = 1/2$$

2. $A \subset B \Rightarrow A^c \cap B = B \setminus A \Rightarrow P(A^c \cap B) = P(B \setminus A)$

$$= P(B) - P(A)$$
$$= 1/2 - 1/3 = 1/6.$$

3. $P(A \cap B) = 1/8$

$$A^c \cap B = B \setminus (A \cap B) \Rightarrow P(A^c \cap B) = P(B \setminus (A \cap B))$$
$$= P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

Ej 3 $P(A^c \cap B^c) = 2/3$

$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - 2/3 = 1/3.$$

Ej 4 $P(C) = 0.25$

$P(D) = 0.45$

$P(C \cap D) = 0.1$

$$P(C^c \cap D) = P(D \setminus (D \cap C))$$
$$= P(D) - P(D \cap C)$$
$$= 0.45 - 0.1 = 0.35$$

Ej 5 $P(A) = 3/8$

$P(B) = 1/2$

$P(A \cap B) = 1/4$

1. $P(A^c) = 1 - P(A) = 1 - 3/8 = 5/8$

$P(B^c) = 1 - P(B) = 1 - 1/2 = 1/2$

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = 5/8$$

3. $P(A^c \cap B^c) = P(A^c) + P(B^c) - P(A^c \cup B^c)$

$$= P(A^c) + P(B^c) - (1 - P(A \cap B))$$
$$= 5/8 + 1/2 - (1 - 1/4)$$
$$= 5/8 + 1/2 - 3/4 = 3/8.$$

4. $P(A^c \cap B) = P(B \setminus (A \cap B)) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$

$P(A \cap B^c) = P(A \setminus (A \cap B)) = P(A) - P(A \cap B) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}.$

Ej 6 $P(A)=0.2$, $P(B)=0.4$ $\quad P(A\cup B)=?$

1. $P(A\cap B)=0.15$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.2+0.4-0.15=0.45$

2. A y B incompatibles

$P(A\cup B)=P(A)+P(B)=0.2+0.4=0.6$

3. $A\subset B$

$P(A\cup B)=P(B)=0.4$

4. $P(B\cap A^c)=0.35$

$P(B\cap A^c)=P(B)-P(A\cap B)=0.4-P(A\cap B)$

$\Rightarrow P(A\cap B)=0.4-0.35=0.05$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)=0.2+0.4-0.05$
$=0.55$

5. $A\subset B^c \Rightarrow$ A y B son incompatibles $\Rightarrow$ Parte 2.

6. $B^c\subset A^c \Rightarrow A\subset B \Rightarrow$ parte 3.

Ej 7 $P(i)=i\alpha \quad i=1,\dots,6.$

1. $$1=\sum_{i=1}^{6}P(i)=\sum_{i=1}^{6}\alpha i=\alpha\sum_{i=1}^{6}i=\alpha\cdot\frac{6\cdot 7}{2}=21\cdot\alpha$$

$\Rightarrow \alpha = 1/21$.

2. $P(5)=5\cdot\alpha=5/21$.

3. $$P(par)=P(2)+P(4)+P(6)=\frac{2+4+6}{21}=12/21=4/7$$

Ej 8 $P(k)=p(1-p)^{k-1} \quad k=1,\dots,9$

$$P(10)=1-\sum_{k=1}^{9}p(1-p)^{k-1}$$
$$=1-p\cdot\frac{1-(1-p)^9}{1-(1-p)}$$
$$=1-p\cdot\frac{1-(1-p)^9}{p}=(1-p)^9$$

Hemos usado que $\displaystyle\sum_{k=0}^{n}x^k=\frac{1-x^{n+1}}{1-x}$.

Ej 9 $P(X=i)=\dfrac{C}{i(i+1)} \quad i\geq 1$

1. $$1=\sum_{i=1}^{\infty}P(X=i)=C\sum_{i=1}^{\infty}\frac{1}{i(i+1)}=C\sum_{i=1}^{\infty}\frac{1}{i}-\frac{1}{i+1}$$
$$=C\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\dots\right]=C\cdot 1$$

$\Rightarrow C=1$.

2. $$P(X=par)=\sum_{k=1}^{\infty}P(X=2k)=\sum_{k=1}^{\infty}\frac{1}{2k(2k+1)}$$
$$=\sum_{k=1}^{\infty}\frac{1}{2k}-\frac{1}{2k+1}=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots$$

Recordar que $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\dots$

Poniendo $x=1$ obtenemos $P(X=par)=1-\ln(2)\simeq 0.307$

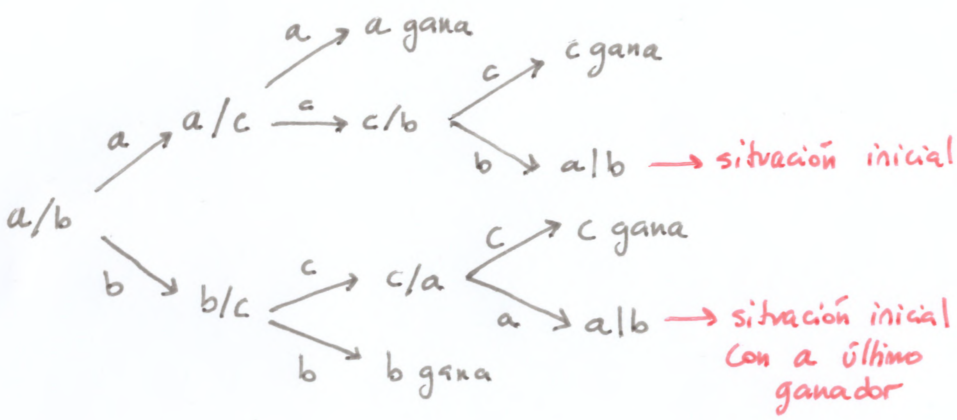
Ej 10 $P(k \text{ ensayos}) = 1/2^k$.

$p = P(a \text{ gana}) \quad q = P(b \text{ gana}) \quad r = P(c \text{ gana})$

$p + q + r = 1$ y por simetría $p = q$.

$\Rightarrow \quad 2p + r = 1$.

1. Hacemos un árbol



Para que gane a debe darse una secuencia del tipo: (1) $aa, (abc)aa, (abc)(abc)aa, \dots$

$\dots, \underbrace{(abc)\cdots(abc)}_{k \text{ veces}}aa$

$$P(\underbrace{(abc)\cdots(abc)}_{k \text{ veces}}aa) = P\left(\begin{array}{l}\text{un juego que}\\ \text{demora } 3k+2 \text{ ensayos}\end{array}\right)$$

$$= \frac{1}{2^{3k+2}}.$$

(2) $(bca)a, (bca)(bca)a, \dots, \underbrace{(bca)\cdots(bca)}_{k \text{ veces}}a$

$$P(\underbrace{(bca)\cdots(bca)}_{k \text{ veces}}a) = \frac{1}{2^{3k+1}}$$

Entonces:

$$P(a \text{ gane}) = \sum_{k=0}^{\infty} \frac{1}{2^{3k+2}} + \sum_{k=1}^{\infty} \frac{1}{2^{3k+1}}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{8^k} + \frac{1}{2}\cdot\frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{8^k}$$

$$= \frac{1}{4}\,\frac{1}{1-1/8} + \frac{1}{2}\cdot\frac{1}{8}\cdot\frac{1}{1-1/8} = \frac{2}{7} + \frac{1}{14} = 5/14$$

De aquí despejamos $r = 2/7$.

4. Probabilidad Condicional

Ej 1 $P(A)=1/2 \quad P(B)=1/3 \quad P(A\cap B)=1/4$

1. $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{1/4}{1/3} = 3/4$

2. $P(B|A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{1/4}{1/2} = 1/2$

3. $P(A^c|B) = 1 - P(A|B) = 1 - 3/4 = 1/4$

4. $P(B^c|A) = 1 - P(B|A) = 1 - 1/2 = 1/2$

5. $P(A^c|B^c) = \dfrac{P(A^c\cap B^c)}{P(B^c)} = \dfrac{1-P(A\cup B)}{1-P(B)} = \dfrac{1-(1/2+1/3-1/4)}{1-1/3}$

   $= \dfrac{5/12}{2/3} = \dfrac{5}{12}\cdot\dfrac{3}{2} = 5/8$

6. $P(B^c|A^c) = \dfrac{P(A^c\cap B^c)}{P(A^c)} = \dfrac{1-P(A\cup B)}{1-P(A)} = \dfrac{5/12}{1/2} = 5/6$

Ej 2 $P(A)=1/4 \quad P(A\cup B)=1/3$

1. $P(B) = P(A\cup B) - P(A) + P(A\cap B)$

   $= 1/3 - 1/4 + P(A)P(B) = 1/3 - 1/4 + 1/3\, P(B)$

   $2/3\, P(B) = 1/12 \Rightarrow P(B) = 1/8$.

2. $P(B) = P(A\cup B) - P(A) = \frac{1}{3} - \frac{1}{4} = 1/12$.

3. $\underbrace{P(A\cup B)}_{1/3} = P(B)$.

Ej 3 $A = \{\text{suma} = 3\} \quad B = \{\text{suma} = 7\}$

$C = \{\text{al menos 1 es 1}\}$

$P(A|C) = \dfrac{P(A\cap C)}{P(C)} \underset{\text{pues } A\subset C}{=} \dfrac{P(A)}{P(C)} = \dfrac{2/36}{1-25/36} = 2/11$

A y C no son independientes.

$P(B|C) = \dfrac{P(B\cap C)}{P(C)} = \dfrac{2/36}{11/36} = 2/11 \neq P(B) = 1/6$

B y C no son independientes.

Ej 4

1. $P(\text{par}) = \frac{10}{20} = 1/2 \qquad P(\leq 10) = \frac{10}{20} = 1/2$

   $P(\text{par y} \leq 10) = P(2,4,6,8,10) = \frac{5}{20} = \frac{1}{4}$

   $\Rightarrow \{\text{par}\}$ y $\{\leq 10\}$ son independientes.

2. $P(\text{primo}) = P(2,3,5,7,11,13,17,19) = \frac{8}{20} = \frac{2}{5}$

   $P(\text{primo y par}) = P(2) = 1/20 \neq P(\text{par})P(\text{primo})$

   $= \frac{1}{2}\cdot\frac{2}{5} = 1/5$.

$\{\text{par}\}$ y $\{\text{primo}\}$ no son independientes.

Ej 5 $P(A) = 0.4 \quad P(B) = 0.3 \quad P((A \cup B)^c) = 0.42$ ③

$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - 0.42 = 0.58$

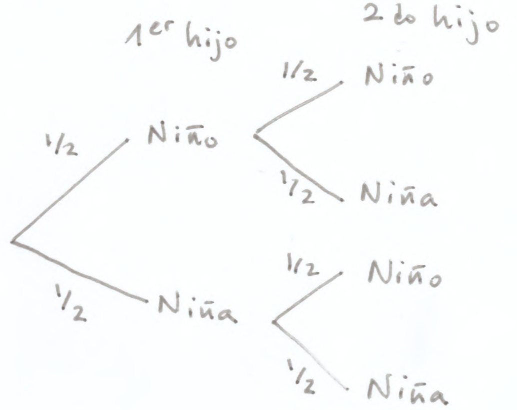
$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12$

$P(A)P(B) = 0.4 \times 0.3 = 0.12 = P(A \cap B)$

A y B son independientes.

Ej 6

1.



$P(2^{do} \text{ niña} \mid 1^{ro} \text{ niña}) = 1/2$

2. $P(\text{ambos niños} \mid \text{al menos 1 niño})$

$$= \frac{P(\text{ambos niños})}{P(\text{al menos 1 niño})} = \frac{1/4}{3/4} = \frac{1}{3}$$

¿Contra intuitivo?

Ej 7 ④

6R 4B 5A

1) $R_1 = \{1^{era} \text{ roja}\} \quad B_2 = \{2^{da} \text{ blanca}\}, \; A_3 = \{3^{ra} \text{ azul}\}$

$P(R_1 \cap B_2 \cap A_3) = \frac{6}{15} \cdot \frac{4}{14} \cdot \frac{5}{13}$

$P(R_1) = 6/15 \quad P(B_2) = 4/15 \quad P(A_3) = 5/15$

$P(R_1 \cap B_2 \cap A_3) \neq P(R_1)P(B_2)P(A_3)$

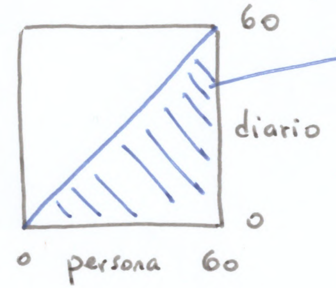
No son independientes.

2) $P(R_1 \cap B_2 \cap A_3) = \frac{6}{15} \cdot \frac{4}{15} \cdot \frac{5}{15} = P(R_1)P(B_2)P(A_3)$

Sí son independientes.

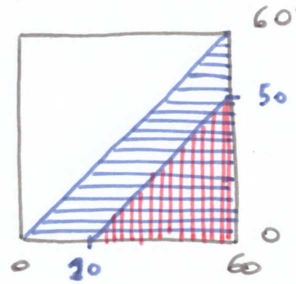
Ej 8

1.



prob. = 1/2

2.

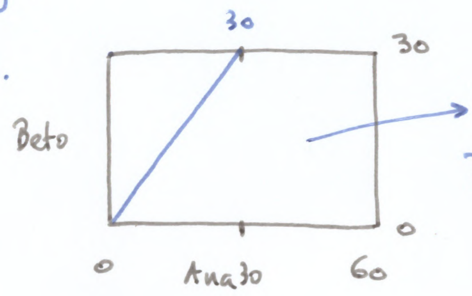


$P(\text{lea completamente} \mid \text{recibe})$

$$= \frac{50^2/2}{60^2/2} = \left(\frac{5}{6}\right)^2.$$

Ej 9

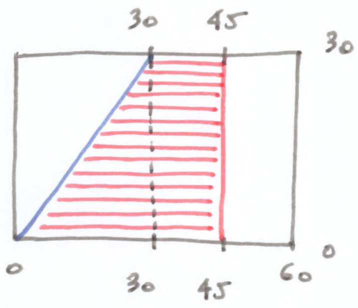
1.



Beto llega antes

$P(\text{Beto antes}) = 3/4$.

2.



$$P(\text{encontrarse} \mid \text{Beto antes}) = \frac{15 \times 30 + \frac{30^2}{2}}{30^2 + 30^2/2} = \frac{30^2}{(1+1/2)\,30^2}$$

$$= 2/3.$$

Ej 10 $H = \{X \text{ culpable}\}$ (hipótesis)

$E = \{X \text{ es tipo } A\}$ (evidencia)

Probabilidades <u>a priori</u> son:

$$P(H) = 1/2 \qquad P(H^c) = 1/2$$

Hay que pensar a E como un evento que <u>a priori</u> puede o no ocurrir. En ese caso

$$P(E \mid H) = 1 \quad \text{y} \quad P(E \mid H^c) = 1/10$$

De la misma forma que en el Ej 3 del Práctico 3, podemos escribir (las probabilidades a posteriori)

$$\frac{P(H \mid E)}{P(H^c \mid E)} = \frac{P(E \mid H)\,P(H)}{P(E \mid H^c)\,P(H^c)}$$

$$= \frac{1 \times 1/2}{1/10 \times 1/2} = 10$$

Como $P(H \mid E) + P(H^c \mid E) = 1$, vemos que

$$P(H \mid E) = \frac{10}{11} \quad \text{y} \quad P(H^c \mid E) = \frac{1}{11}.$$

<u>Conclusión</u>:

Dada la nueva evidencia de que X es tipo A, el evento E, la probabilidad de que X sea culpable (evento H) es $10/11$.