13. Teorema Central del Límite

Ej 1 $X \sim N(0,1)$

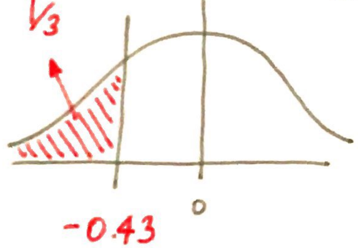
$P(0 \leq X \leq 0.001) = \varphi(0) \times 0.001 = \frac{1}{\sqrt{2\pi}} \cdot 0.001$

$\simeq 4 \times 10^{-4}$

$P(1 \leq X \leq 1.001) = \varphi(1) \times 0.001 = 2.4 \times 10^{-4}$

Ej 2 $X \sim N(\mu, \sigma^2)$

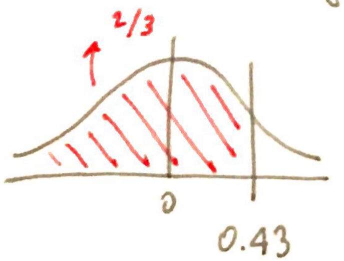
$P(X \leq 0) = P\left(\frac{X-\mu}{\sigma} \leq \frac{-\mu}{\sigma}\right) = \Phi\left(\frac{-\mu}{\sigma}\right) = \frac{1}{3}$



$\Rightarrow \frac{-\mu}{\sigma} = -0.43$

$\boxed{\mu = 0.43\sigma}$

$P(X \leq 1) = P\left(\frac{X-\mu}{\sigma} \leq \frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1-\mu}{\sigma}\right) = \frac{2}{3}$

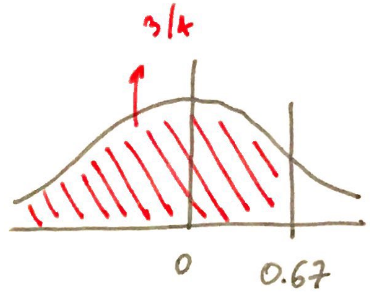


$\frac{1-\mu}{\sigma} = 0.43$

$\boxed{1-\mu = 0.43\sigma}$

$\mu = 1-\mu \Rightarrow \mu = 1/2 \qquad \sigma = \frac{0.5}{0.43} = 1.16$

Si $P(X \leq 1) = P\left(\frac{X-\mu}{\sigma} = \frac{1-\mu}{\sigma}\right) = \Phi\left(\frac{1-\mu}{\sigma}\right) = \frac{3}{4}$



$\frac{1-\mu}{\sigma} = 0.67$

$1-\mu = 0.67\sigma$

$\mu = 1 - 0.67\sigma$

$0.43\sigma = 1 - 0.67\sigma \Rightarrow \sigma(0.43+0.67) = 1$

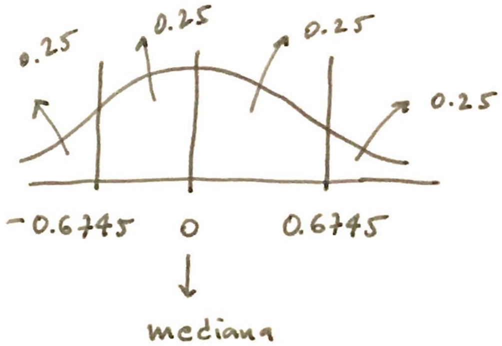
$\Rightarrow \sigma = 1/1.1 \simeq 0.9 \Rightarrow \mu = 0.39.$

Ej 3 $X \sim N(\mu, \sigma^2) \qquad Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

1. $P(X \in [\mu - k\sigma, \mu + k\sigma]) = P(Z \in [-k,k])$

k	1	2	3
$P(Z \in [-k,k])$	0.6827	0.9545	0.9973

2.



Ej 4 $X \sim N(\mu, \sigma^2)$ $\mu = 280$ $\sigma = 8.5$

1. La fecha de inicio del embarazo es 25 julio − 280 días = D.

   Entonces $\underbrace{18 \text{ julio} - 25 \text{ julio}}_{-7} + \underbrace{25 \text{ julio} - D}_{280} + D$

   $\Rightarrow$ 18 julio = 273 después de D.

   $$P(X < 273) = P\left(\frac{X-280}{8.5} < \frac{273-280}{8.5}\right)$$

   $$= \Phi\left(\frac{-7}{8.5}\right) = 0.205 \quad (\text{casi } 1 \text{ en } 5)$$

2. Queremos la prob. de que nazca entre 19 julio $(X-280=-6)$ 31 de julio $(X-280=6)$.

   $$P(-6 \le X-280 \le 6) = \Phi\left(\frac{6}{8.5}\right) - \Phi\left(\frac{-6}{8.5}\right)$$

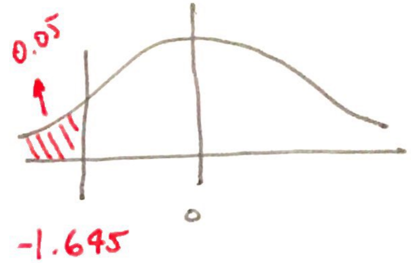
   $$= 0.520$$

3. Queremos encontrar x tal que

   $$P(X \ge x) = 0.95 \Rightarrow P\left(\frac{X-280}{8.5} \ge \frac{x-280}{8.5}\right) = 0.95$$

   $$\Rightarrow 1 - \Phi\left(\frac{x-280}{8.5}\right) = 0.95$$

$$\Phi\left(\frac{x-280}{8.5}\right) = 0.05$$



Entonces

$$\frac{x-280}{8.5} = -1.645$$

$$\Rightarrow x = 280 - 1.645 \times 8.5 \simeq 280 - 14$$

Debe fijarlo 14 días antes de 25 de julio

$\rightarrow$ el 11 de julio (como máximo).

Ej 5 Usaremos la regla de que

$$P(\mu - 3\sigma \le X \le \mu + 3\sigma) = 0.997$$

(casi toda el área)

Curva 1: casi toda el área está entre −1.5 y 1.5

$\Rightarrow \frac{3}{2} = 3\sigma \Rightarrow \sigma = 1/2.$

Curva 2: casi toda el área está entre −3 y 3

$\Rightarrow 3 = 3\sigma \Rightarrow \sigma = 1.$

Curva 3: casi toda el área está entre −6 y 6

$\rightarrow 3\sigma = 6 \Rightarrow \sigma = 2.$

Ej 6 $Z \sim N(0,1)$

a)

$$P(|Z| \leq z) = P(-z \leq Z \leq z)$$

$$= \Phi(z) - \Phi(-z) = F_{|Z|}(z)$$

Derivando:

$$p_{|Z|}(z) = 2\varphi(z) \quad (z > 0).$$

b) $Y = Z^2$ $\quad \frac{dy}{dz} = 2z$ $\quad$ (hay 2 preimágenes)

$$p_{Z^2}(y) = \frac{1}{2z} p_Z(z) = \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$p_{Z^2}(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \quad (y > 0).$$

c) $Y = \frac{1}{Z}$ $\quad \frac{dy}{dz} = \frac{-1}{z^2}$ $\quad$ (hay una preimagen)

$$p_Y(y) = \frac{1}{|-1/z^2|} p_Z(z) = z^2 \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$= \frac{1}{\sqrt{2\pi}\, y^2} e^{-\frac{1}{2y^2}} \quad (y \neq 0).$$

d) $Y = 1/Z^2$ $\quad \frac{dy}{dz} = \frac{-2}{z^3}$ $\quad$ (hay dos preimágenes $1/\sqrt{y}$ $\;-1/\sqrt{y}$)

$$p_Y(y) = \frac{1}{2\, y^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-1/2y} \times 2 \quad (y > 0)$$

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{y^{3/2}} e^{-1/2y} \quad (y > 0).$$

Ej 7 $X \sim Bin(100, 1/3)$ $\quad \mu = \frac{100}{3}$ $\quad \sigma^2 = \frac{100}{3} \cdot \frac{2}{3}$

$$P(X \leq 30) = P\left(\frac{X - 100/3}{\sqrt{\frac{100}{3} \cdot 2/3}} \leq \frac{30 - 100/3}{\sqrt{\frac{100}{3} \cdot 2/3}}\right)$$

$$\cong \Phi(-0.707) = 0.2398.$$

Ej 8 $X_1, \dots, X_{81}$ iid $\quad \mu = E(X_i) = 5$ $\quad \sigma^2 = Var(X_i) = 4$

$$P(X_1 + \dots + X_{81} > 369) = P(S_{81} > 369)$$

$$= P\left(\frac{S_{81} - 81 \cdot 5}{\sqrt{81} \cdot 2} > \frac{369 - 81 \cdot 5}{\sqrt{81} \cdot 2}\right)$$

$$\underset{S_{81} \approx \text{Normal}}{\cong} 1 - \Phi\left(\frac{369 - 81 \cdot 5}{\sqrt{81} \cdot 2}\right) = 1 - \Phi\left(\frac{-36}{18}\right)$$

$$= 1 - \Phi(-2) = \Phi(2) = 0.9772$$

Ej 9 $n = 100$ pedidos $\quad X_i$ = precio de i-ésimo pedido

$R(X_i)$ redondeo de X_i al múltiplo de 5 más cercano.

E_i = error de redondeo del pedido $i = X_i - R(X_i)$

$$E = \sum_{i=1}^{100} E_i = \text{error total de redondeo.}$$

Queremos estimar

$$p = P(|E| > 50)$$

Vamos a suponer que el error de redondeo E_i tiene distribución uniforme en $[-2.5, 2.5]$

$$\Rightarrow \underset{\mu}{\underset{\shortparallel}{E(E_i)}} = 0 \quad y \quad \underset{\sigma^2}{\underset{\shortparallel}{Var(E_i)}} = \frac{5^2}{12} = \frac{25}{12} = 2.083$$

$\sigma = 1.44.$

Entonces $E \overset{d}{\approx} N(0, 100\sigma^2)$

$$P(|E| > 50) = P\left(\left|\frac{E}{10\sigma}\right| > \frac{50}{10\sigma}\right) = P\left(\left|\frac{E}{10\sigma}\right| > 3.47\right)$$

$$= 2(1 - \Phi(3.47)) = 0.0005.$$

Ej 10 0.5 A, 0.2 B, 0.3 C, D, E.

$n = 400$

1. $A_i = \begin{cases} 1 & \text{si vota } A \\ 0 & \text{si no} \end{cases} \sim Ber(0.5)$

$$\overline{A}_{400} \sim \frac{1}{400} Bin(400, 0.5) \approx N\left(0.5, \frac{1}{400 \cdot 4}\right)$$

$$P(\overline{A}_{400} \geq 0.52) = P\left(\frac{\overline{A}_{400} - 0.5}{1/40} \geq \frac{0.52 - 0.5}{1/40}\right)$$

$$= P\left(\frac{\overline{A}_{400} - 0.5}{1/40} \geq 0.8\right) = 1 - \Phi(0.8)$$

$$= 0.2119$$

2. O = Otros = C, D, E.

$O_i = \begin{cases} 1 & \text{si } O \\ 0 & \text{si no} \end{cases} \sim Ber(0.3)$

$$\overline{O}_{400} \overset{d}{\approx} N\left(0.3, \frac{0.3 \times 0.7}{400}\right) = N(0.3, (0.023)^2)$$

$$P(\overline{O}_{400} \leq 0.25) = P\left(\frac{\overline{O}_{400} - 0.3}{0.023} \leq \frac{0.25 - 0.3}{0.023}\right)$$

$$= \Phi(-2.17) = 0.015.$$