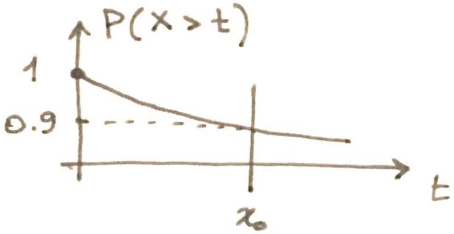
11. Variables aleatorias continuas II

1

Ej 1 $X \sim Exp(0.01)$

$$P(X > x_0) = e^{-\lambda x_0} = e^{-0.01 x_0} = 0.9$$

$$-0.01\, x_0 = \ln 0.9 \Rightarrow x_0 = \frac{\ln 0.9}{-0.01} = 10.54$$



Ej 2 $T \sim Exp(1/8)$

$$P(T > 8) = e^{-\lambda 8} = e^{-1}$$

$T_1, T_2, \dots, T_8$ independientes.

$$X_i = \begin{cases} 1 & \text{si } T_i > 8 \\ 0 & \text{si no.} \end{cases} \Rightarrow X_i \sim Ber(e^{-1})$$

$S = X_1 + \dots + X_8$ componentes funcionando después de 8 años.

$S \sim Bin(8, e^{-1})$

$$P(S \geq 2) = 1 - P(S < 2) = P(S=0) + P(S=1)$$
$$= (1-e^{-1})^8 + 8e^{-1}(1-e^{-1}) = 0.856$$

Ej 3

2

$N_t \sim Poiss(\lambda t) \quad \lambda > 0$

X = tiempo de espera.

1. Notar que $\{X > t\}$ si no pasa ningún omnibus en $[0, t]$.
   Entonces
   $$P(X > t) = P(N_t = 0) = e^{-\lambda t}.$$

2. La densidad de X es la derivada de la fda $F(t) = 1 - e^{-\lambda t}$
   $$p(t) = F'(t) = \lambda e^{-\lambda t}.$$
   $\Rightarrow X \sim Exp(\lambda)$.

3. $E(X) = \frac{1}{\lambda}$.

4. $\lambda = 1/15$. $P(X \leq 1/15) = 1 - e^{-\frac{1}{15} \cdot 15}$
   $= 1 - e^{-1} = 0.632$

Ej 4 $X_1, X_2 \sim Exp(\lambda)$

1. $T = \min\{X_1, X_2\}$

$$P(T>t) = P(\min\{X_1,X_2\}>t)$$
$$= P(X_1>t, X_2>t)$$
$$= P(X_1>t)P(X_2>t)$$
$$= e^{-\lambda t} e^{-\lambda t} = e^{-2\lambda t}$$

$$\Rightarrow F(t) = P(T \leq t) = 1 - e^{-2\lambda t}$$

$$\Rightarrow T \sim Exp(2\lambda).$$

2. $B_1 \sim Exp(2)$, $B_2 \sim Exp(3)$, $B_3 \sim Exp(5)$

$T = \min\{B_1, B_2, B_3\}$

$$P(T>t) = P(B_1>t)P(B_2>t)P(B_3>t)$$
$$= e^{-2t} e^{-3t} e^{-5t}$$
$$= e^{-10t} \Rightarrow T \sim Exp(10)$$

$$\Rightarrow E(T) = \frac{1}{10}.$$

Ej 5 $X \sim exp(\lambda)$

1. $cX = Y \qquad y = cx = g(x) \qquad \frac{dy}{dx} = c$

$$p_Y(y) = \frac{1}{c} p_X(x) = \frac{1}{c} p_X(y/c) \quad (y>0)$$

$$p_Y(y) = \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} \quad (y>0) \Rightarrow Y \sim exp\left(\frac{\lambda}{c}\right).$$

2. $Z = a e^X \qquad z = a e^x \qquad \frac{dz}{dx} = a e^x$

$$p_Z(z) = \frac{1}{a e^x} p_X(x) = \frac{1}{a e^x} \lambda e^{-\lambda x}$$

$$= \lambda \frac{1}{(a e^x)^{\lambda+1}} a^\lambda = \lambda a^\lambda \, 1/z^{\lambda+1}$$
$$(z>a)$$

$$p_Z(z) = \lambda a^\lambda \frac{1}{z^{\lambda+1}} \quad (z>a).$$

$$P(Z \leq z) = \int_a^z \lambda a^\lambda \frac{1}{u^{\lambda+1}} du = \lambda a^\lambda \left.\frac{u^{-\lambda}}{-\lambda}\right|_a^z$$

$$= 1 - \left(\frac{a}{z}\right)^\lambda \quad (z>a)$$

$$F(z) = \begin{cases} 0 & si \quad z<a \\ 1 - \left(\frac{a}{z}\right)^\lambda & si \quad z \geq a. \end{cases}$$

(5)

3. $Y = X^2 \qquad y = x^2 \qquad \frac{dy}{dx} = 2x$

$$p_Y(y) = \frac{1}{2x} p_X(x) = \frac{1}{2\sqrt{y}} \lambda e^{-\lambda\sqrt{y}} \quad (y>0).$$

X e y son ≥ 0.

$$P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$$

$$= 1 - e^{-\lambda\sqrt{y}}$$

$$F(y) = \begin{cases} 0 & \text{si } y<0 \\ 1-e^{-\lambda\sqrt{y}} & \text{si } y \geq 0. \end{cases}$$

Ej 6 $\quad Y = X^2 \qquad y = x^2 \qquad \frac{dy}{dx} = 2x$

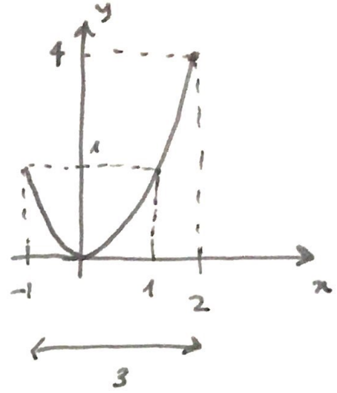
1. $y = x^2$ es inyectiva en $(0,1)$

$$p_Y(y) = \frac{1}{2x} p_X(x) = \begin{cases} 0 & \text{si } y \notin (0,1) \\ \frac{1}{2\sqrt{y}} & \text{si } y \in (0,1). \end{cases}$$

2. $$p_Y(y) = \sum_{y=x^2} \frac{1}{|2x|} p_X(x) = \begin{cases} 0 & \text{si } y \notin (0,1) \\ \frac{1}{2\sqrt{y}} \cdot \frac{1}{2} + \frac{1}{2\sqrt{y}} \cdot \frac{1}{2} & y \in (0,1) \end{cases}$$

$$= \begin{cases} 0 & \text{si } y \in (0,1) \\ \frac{1}{2\sqrt{y}} & \text{si } y \in (0,1) \end{cases}$$ igual que antes.

(6)



Si $y \in (0,1)$: hay dos preimagenes

$$p_Y(y) = \frac{1}{2\sqrt{y}} \cdot \frac{1}{3} + \frac{1}{2\sqrt{y}} \cdot \frac{1}{3} = \frac{1}{3\sqrt{y}}$$

Si $y \in (0,2)$: hay una sola

$$p_Y(y) = \frac{1}{2\sqrt{y}} \cdot \frac{1}{3} = \frac{1}{6\sqrt{y}}$$

$$p_Y(y) = \begin{cases} 0 & \text{si } y \notin (0,2) \\ \frac{1}{3\sqrt{y}} & \text{si } y \in (0,1) \\ \frac{1}{6\sqrt{y}} & \text{si } y \in (1,2) \end{cases}$$

Ej 7 $\quad X \sim \mathcal{C}(0,1) \qquad p_X(x) = \frac{1}{\pi(1+x^2)} \qquad x \in \mathbb{R}$

$$Y = \frac{1}{X}, \quad y = \frac{1}{x} \quad \frac{dy}{dx} = \frac{-1}{x^2}$$

$$p_Y(y) = \frac{1}{\left|{-1}/{x^2}\right|} p_X(x) = \frac{x^2}{\pi} \frac{1}{1+x^2} = \frac{1}{y^2\pi} \frac{1}{1+\frac{1}{y^2}}$$

$$\Rightarrow p_Y(y) = \frac{1}{\pi(1+y^2)} \Rightarrow Y \sim \mathcal{C}(0,1)$$

también

¿sorprendente?

Ej 8 $\quad \rho(x,y) = c\,(x^2+xy)$ en $[0,1)^2$

1. $$\int_0^1\int_0^1 c\,(x^2+xy)\,dx\,dy$$

$$= c\int_0^1 \left[\frac{x^3}{3}+\frac{x^2y}{2}\right]_0^1 dy$$

$$= c\int_0^1 \frac{y}{2}+\frac{1}{3}\,dy = c\left[\frac{y^2}{4}+\frac{y}{3}\right]_0^1 = c\left(\frac{1}{4}+\frac{1}{3}\right)$$

$$= c\,\frac{7}{12} = 1 \Rightarrow c = \frac{12}{7}$$

2. $$p(x) = \int_0^1 \rho(x,y)\,dy = c\int_0^1 x^2+xy\,dy$$

$$= c\left(x^2 + x\frac{y^2}{2}\Big|_0^1\right) = c\left(x^2+\frac{x}{2}\right)$$

$$F_X(x) = \int_0^x c\left(u^2+\frac{u}{2}\right)du = c\left(\frac{u^3}{3}+\frac{u^2}{4}\right)_0^x$$

$$= \frac{12}{7}\left(\frac{x^3}{3}+\frac{x^2}{4}\right) \quad x \text{ en } [0,1].$$

Para Y podemos hacer lo mismo, pero calcularemos primero F_Y.

$$F_Y(y) = P(Y\le y) = \int_0^y\int_0^1 c\,(x^2+xu)\,dx\,du$$

$$= \int_0^y c\left(\frac{x^3}{3}+\frac{x^2u}{2}\right)_0^1 du$$

$$= \int_0^y c\left(\frac{1}{3}+\frac{u}{2}\right)du = c\left(\frac{1}{3}y+\frac{y^2}{4}\right)$$

$$F_Y(y) = \frac{12}{7}\left(\frac{y}{3}+\frac{y^2}{4}\right) \quad y\in[0,1]$$

3. Derivando

$$p_Y(y) = F_Y'(y) = \frac{12}{7}\left(\frac{1}{3}+\frac{y}{2}\right) \quad y\in[0,1].$$

$p_X(y)$ fue calculada en 2.

4. $$E(X) = \int_0^1 c\,x\left(\frac{x^3}{3}+\frac{x^2}{4}\right)dx$$

$$= c\int_0^1 \frac{x^4}{3}+\frac{x^3}{4}\,dx = c\left[\frac{x^5}{3\cdot5}+\frac{x^4}{4\cdot4}\right]_0^1$$

$$= c\left[\frac{1}{3\cdot5}+\frac{1}{4\cdot4}\right] = \frac{31}{140} = 0.22$$

$$E(X^2) = \int_0^1 c\left(\frac{x^5}{3}+\frac{x^4}{4}\right)dx = \frac{19}{105} \approx 0.18$$

$\mathrm{Var}(X) = 0.1316.$

4.

$$E(XY) = \int_0^1 \int_0^1 c\, xy\,(x^2 + xy)\,dx\,dy$$

$$= \frac{17}{42} = 0.40$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = 0.40 - 0.22 \cdot \underbrace{0.57}_{E(Y)}.$$

$$= 0.27$$

Ej 9 $p(x,y) = c\, x^2 y(1+y)$ $\quad 0 \le x \le 3$, $0 \le y \le 3$

1. $$c \int_0^3 \int_0^3 x^2 y(1+y)\,dx\,dy = c \cdot \frac{243}{2} = 1$$

$$\Rightarrow c = 2/243.$$

2. $$P(1 \le X \le 2, 0 \le Y \le 1) = \int_0^1 \int_1^2 p(x,y)\,dx\,dy$$

$$= \frac{35}{2187} = 0.016$$

3. $$p_X(x) = \int_0^3 p(x,y)\,dy = \frac{x^2}{9}$$

$$F_X(x) = x^3/27$$

4. Claramente $\left(\frac{x^3}{27}\right)' = x^2/9$.

5. Sí, ya que $p_Y(y) = \frac{2}{27} y(y+1)$

y $p_X(x) = \frac{x^2}{9}$ de donde

$$p(x,y) = p_X(x)\, p_Y(y).$$

Ej 10 $p(x,y) = x+y$ en $[0,1]\times[0,1]$.

1. $$p(x) = \int_0^1 (x+y)\,dy = x + \frac{1}{2} \quad x \text{ en } [0,1]$$

$$p(y) = \int_0^1 (x+y)\,dx = y + \frac{1}{2} \quad y \text{ en } [0,1]$$

2. No ya que $p(x)p(y) = (x+1/2)(y+1/2)$

$$\neq x + y = p(x,y).$$

3. $E(X) = \frac{7}{12}$, $E(Y) = \frac{7}{12}$, $E(X^2) = \frac{5}{12} = E(Y^2)$

$$E(X^2+Y^2) = 10/12.$$

$$\mathrm{Cov}(X,Y) = \underbrace{E(XY)} - E(X)E(Y)$$

$$= 1/3 - \left(\frac{7}{12}\right)^2 = \frac{-1}{144}.$$

12. Ley de los grandes números

Ej 1 $p = 1/2 \quad n = 100 \qquad np = 50 = \mu = E(S_n)$

$S_n = \text{caras} \qquad np(1-p) = 25 = \sigma^2 = Var(S_n)$

$$P(|S_n - \mu| > 3\sigma) \leq \frac{Var(S_n)}{(3\sigma)^2} = \frac{\sigma^2}{9\sigma^2} = 1/9$$

Ej 2 $X = \text{puntaje} \qquad \mu = E(X) = 70$

$\sigma^2 = Var(X) = 25$

1.

$$P(X \in [65, 75]) = P(|X - 70| \leq 5)$$

$$= 1 - P(|X - 70| > 5) = 1 - P(|X - \mu| > \sigma)$$

$$\underset{\text{Chebyshev}}{\geq} 1 - \frac{Var(X)}{\sigma^2} = 1 - \frac{\sigma^2}{\sigma^2} = 0$$

No se puede decir nada.

2. $n = 100$ promedio de la clase $= \overline{X}_{100}$

$E(\overline{X}_{100}) = \mu = 70$

$Var(\overline{X}_{100}) = \frac{\sigma^2}{100} = \frac{25}{100}$

$$P(\overline{X}_{100} \in [65, 75])$$

$$= 1 - P(|\overline{X}_{100} - \mu| > \sigma)$$

$$\geq 1 - \frac{Var(\overline{X}_{100})}{\sigma^2} = 1 - \frac{\sigma^2}{100\sigma^2} = \frac{99}{100}.$$

Ej 3 $Y_n = \text{precio día } n$

$X_n = Y_{n+1} - Y_n = \text{cambio de precio}$

$E(X_n) = 0, \quad Var(X_n) = \sigma^2 = 1/4$

$Y_1 = 30$

1. $P(25 \leq Y_2 \leq 35) = P(25 \leq Y_1 + X_2 \leq 35)$

$$= P(25 \leq 30 + X_2 \leq 35)$$

$$= P(-5 \leq X_2 \leq 5) = P(|X_2| \leq 5)$$

$$= 1 - P(|X_2| > 5) \geq 1 - \frac{Var(X_2)}{5^2}$$

$$= 1 - \frac{1}{4 \cdot 25} = 1 - \frac{1}{100} = 99/100.$$

2. $P(25 \leq Y_{11} \leq 35) = P(25 \leq X_{11} + X_{10} + \cdots + X_2 + Y_1 \leq 35)$

$$= P(|X_{11} + \cdots + X_2| \geq 5)$$

$$= 1 - P(|X_{11} + \cdots + X_2| < 5)$$

$$= 1 - \frac{Var(X_{11} + \cdots + X_2)}{25} = 1 - 10 \cdot \frac{1}{4 \cdot 25} = 1 - \frac{10}{100}$$

$$= 90/100$$

3. $P(25 \leq Y_{101} \leq 35) = P\left(25 \leq Y_1 + \sum_{j=2}^{101} X_j \leq 35\right)$

$$= P\left(\left|\sum_{j=2}^{101} X_j\right| \leq 5\right) = 1 - P\left(\left|\sum_{j=2}^{101} X_j\right| > 5\right) \geq$$

$$\geq 1 - \frac{\mathrm{var}\left(\sum_{j=2}^{101} X_j\right)}{25} = 1 - 100 \cdot \frac{1}{4 \cdot 25} = 0.$$

Ej 4 $X \sim \exp(1)$

$$P(X \geq x) = e^{-x}$$

Markov: $P(X \geq x) \leq \frac{E(X)}{x} = \frac{1}{x}$

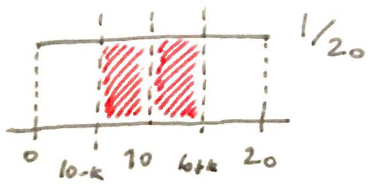
x	0.5	1	2
$P(X \geq x)$	0.6065	0.3679	0.1353
Markov	2	1	0.5

Ej 5 $\mu = E(X) = 10$, $\sigma^2 = 100/3$

$$P(|X-10| \geq k) \leq \frac{Var(X)}{k^2} = \frac{100}{3k^2}$$

k	2	5	9	20
$P(\lvert X-10\rvert \geq k)$ $\leq$	8.3	1.3	0.41	0.08

Ej 6 $E(X) = 10$ $Var(X) = 100/3$



$$P(|X-10| > k) = 2 \cdot \frac{10-k}{20} = 10-k/10$$

k	2	5	9	20
$P(\lvert X-10\rvert > k)$	0.8	0.5	0.1	0

Ej 7 $\mu = E(X) = 0$ $\sigma^2 = Var(X) = 1$

$$P(|X| \geq k) \leq \frac{Var(X)}{k^2} = \frac{1}{k^2} \leq 0.01 = \frac{1}{10^2}$$

$\Rightarrow k = 10.$

Ej 8 $X \sim Poiss(1)$ $\mu = E(X) = 1$, $var(X) = 1$

$$P(X_1 + \cdots + X_{10} > 15) \leq \frac{E(X_1 + \cdots + X_{10})}{15} = \frac{10}{15} = \frac{2}{3}.$$

$X_1 + \cdots + X_{10} \sim Poiss(10)$

$$P(X_1 + \cdots + X_{10} > 15) = \sum_{j=0}^{14} \frac{10^j}{j!} e^{-10}$$

$$= e^{-10} \sum_{j=0}^{14} \frac{10^j}{j!} = 0.083$$

Ej 9 g es continua en μ

Dado $\varepsilon > 0$, $\exists\ \delta > 0$ t.q. $|g(x) - g(\mu)| \leq \varepsilon$ si $|x - \mu| \leq \delta$.

Dicho de otro modo: si $|g(x) - g(\mu)| > \varepsilon$

entonces $|x-\mu| > \delta$.

$$P\left(|g(\overline{X}_n)-g(\mu)|>\varepsilon\right) \leq P\left(|\overline{X}_n-\mu|>\delta\right) \xrightarrow[n]{} 0$$

Ej 10

Notar que $\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - \overline{X}_n^2$

Por la LGN: $\frac{1}{n}\sum_{i=1}^{n}X_i^2 \xrightarrow{P} E(X^2)$

y $\overline{X}_n \xrightarrow{P} E(X)$

por el ej. anterior $(g(x)=x^2)$ $\overline{X}_n^2 \xrightarrow{P} E(X)^2$

Entonces

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2 \xrightarrow{P} E(X^2)-E(X)^2 = \sigma^2.$$