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Práctico 3

(d) $\begin{cases} x' = -x + y = -1 \cdot x + 1 \cdot y \\ y' = -y = 0 \cdot x - 1 \cdot y \end{cases}$

- $\mathcal{X} : I \subset \mathbb{R} \to \mathbb{R}^2$

$$\mathcal{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\dot{\mathcal{X}} = \boxed{\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}}^{A} \mathcal{X} \to \dot{\mathcal{X}} = A\mathcal{X}$$

- ¿Cómo hacemos para saber si $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$ es diagonalizable?

- $(\lambda - 1)^2 \to \lambda = 1$ es valor propio con $ma = 2$
- $\dim(\mathrm{Ker}(A - \lambda I)) = m_g$

- $ma \geq m_g \geq 1$

En $\mathbb{R}^2$: Una manera rápida de saber si $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$ es diagonalizable es viendo si sus valores propios distintos.

- $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix}$

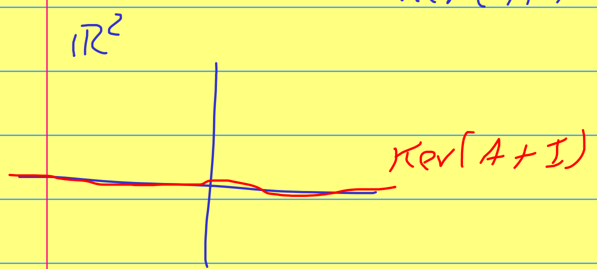
$$p(\lambda) = (-1 - \lambda)^2$$

Entonces el único valor propio es $\lambda = -1$ y tiene $ma(-1) = 2$.

$$\overset{\lambda=-1}{Ker(A+I)} = \left\{(x,y) \in \mathbb{R}^2 : (A+I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{array}{l} \boxed{y=0} \\ 0=0 \end{array} \right.$$

$$Ker(A+I) = \{(x,y) \in \mathbb{R}^2 : y = 0\}$$



$$\Rightarrow m_g(-1) = \dim(Ker(A+I)) = 1$$

Como $m_g(-1) < m_a(-1)$ la matriz A no es diagonalizable, entonces

$$PAP^{-1} = J$$

donde $J = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$

· ¿Quién es $P \in \mathcal{M}_{2x2}(\mathbb{R})$?

Sean v_1, v_2 vectores propios entonces $P = (v_1 | v_2)$

$v_2 \in Ker(A+I)$ | $v_1 \in Ker(A-\lambda I)$

$v_1 = (A+I)v_2$ | $v_2 = (A-\lambda I)v_1$

$$Ker(A+I) = \{y=0\} \rightarrow \boxed{v_2 = (1,0)}$$

$$Av_1 = \lambda v_1 + v_2 \rightarrow \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} -x+y \\ -y \end{pmatrix} = \begin{pmatrix} -x+1 \\ -1 \end{pmatrix}$$

$$-y = -1 \Rightarrow y = 1 \rightarrow \boxed{y=1} \rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$-x+1 = -x+1 \rightarrow \boxed{x=0}$$

$$V1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \boxed{P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\boxed{PAP^{-1} = J}$$

- $J = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{cases} u' = -u \\ v' = u - v \end{cases}$

$$u(t) = Ae^{-t}$$
$$v'(t) = Ae^{-t} - v \rightarrow \boxed{v' + v = Ae^{-t}}$$

$$v_H(t) = Be^{-t}$$

• $v_p(t)$: $v_p(t) = D(t)e^{-t}$

$$v_p' = D'e^{-t} - De^{-t} \rightarrow v_p' - v_p = \underbrace{D'e^{-t} = Ae^{-t}}$$

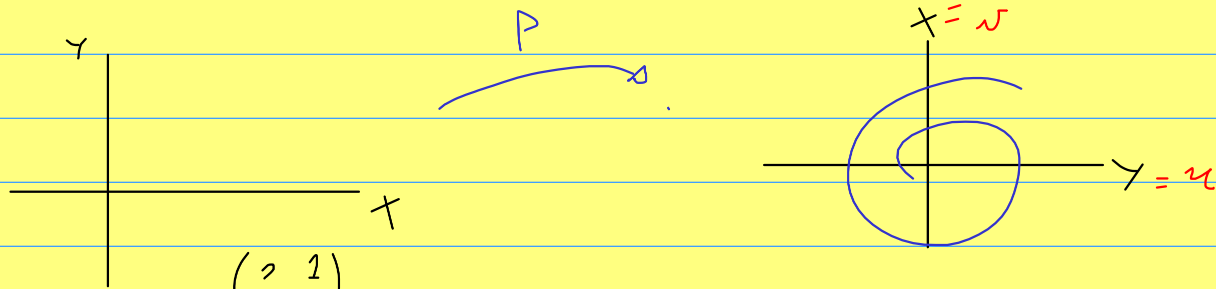
$$\hookrightarrow D'\cancel{e^{-t}} = A\cancel{e^{-t}}$$
$$D' = A \rightarrow \boxed{D = At}$$

$$v_p(t) = Ate^{-t}$$

$$v(t) = Be^{-t} + Ate^{-t}$$

$u(t) = Ae^{-t}$

$v(t) = Be^{-t} + Ate^{-t}$



$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\boxed{X(t) = P \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}} = \begin{pmatrix} v(t) \\ u(t) \end{pmatrix}$

$\to x(t) = Be^{-t} + Ate^{-t}$

$y(t) = Ae^{-t}$

$$\begin{cases} x' = -x + y \\ y' = -y \end{cases}$$

3. (a) Se considera el sistema:

$$\begin{cases} x' = -y \\ y' = x \end{cases} \to A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to p(\lambda) = \lambda^2 - 1$$

$\to$ Las raíces son $\pm i$

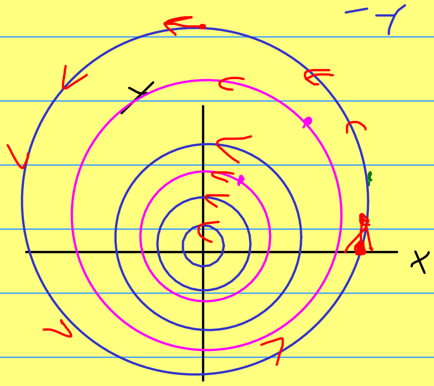
i) Probar que si $\varphi(t) = (x(t), y(t))$ es solución de la ecuación entonces $x^2(t) + y^2(t) = cte.$

ii) A partir de i), dibujar el diagrama de fase.

i) $x^2 + y^2 = cte \iff (x^2+y^2)' = 0$

$$(x^2+y^2)' = 2x\underbrace{x'}_{-y} + 2y\underbrace{y'}_{x} = -2xy + 2xy = 0$$

ii)



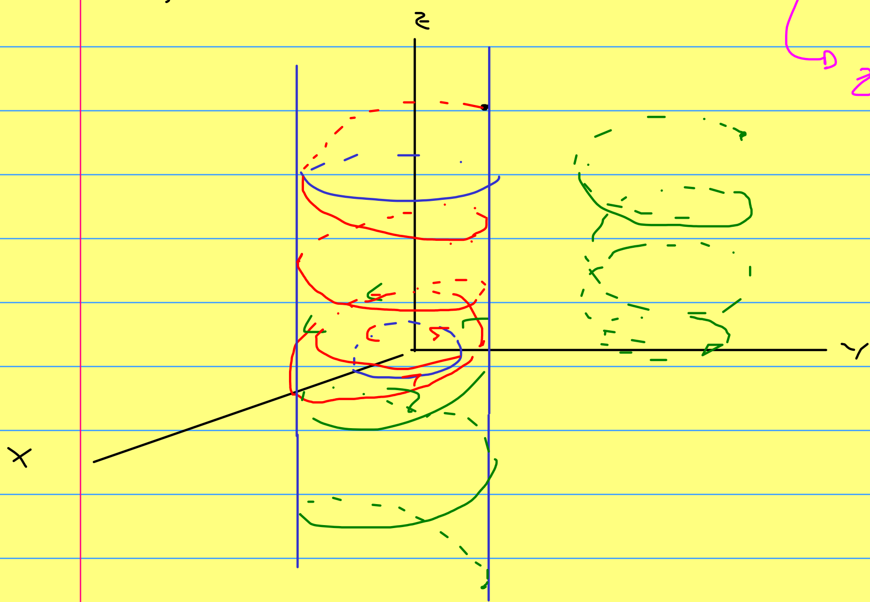
$\varphi(t) = (x(t), y(t))$

$$\boxed{\begin{cases} x' = -y \\ y' = x \end{cases}}$$

(b) A partir de la parte (a), dibujar el diagrama de fase del sistema lineal:

$$\begin{cases} x' = -y \\ y' = x \\ z' = -z \end{cases}$$

$z' = -z \rightarrow z(t) = Ae^{-t}$



(f) $\begin{cases} x' = -x = -1\cdot x + 0\cdot y + 0\cdot z \\ y' = -y = 0\cdot x - 1\cdot y + 0\cdot z \\ z' = z = 0\cdot x + 0\cdot y + 1\cdot z \end{cases} \rightarrow A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$x' = -x \rightarrow x(t) = A e^{-t}$

$y' = -y \rightarrow y(t) = B e^{-t}$

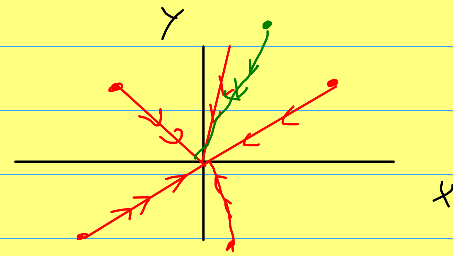
$z' = z \rightarrow \boxed{z(t) = C e^{t}}$

$-t = \ln\left(\frac{x}{A}\right)$

$\rightarrow y = B e^{\ln\left(\frac{x}{A}\right)} = \frac{B}{A} x$

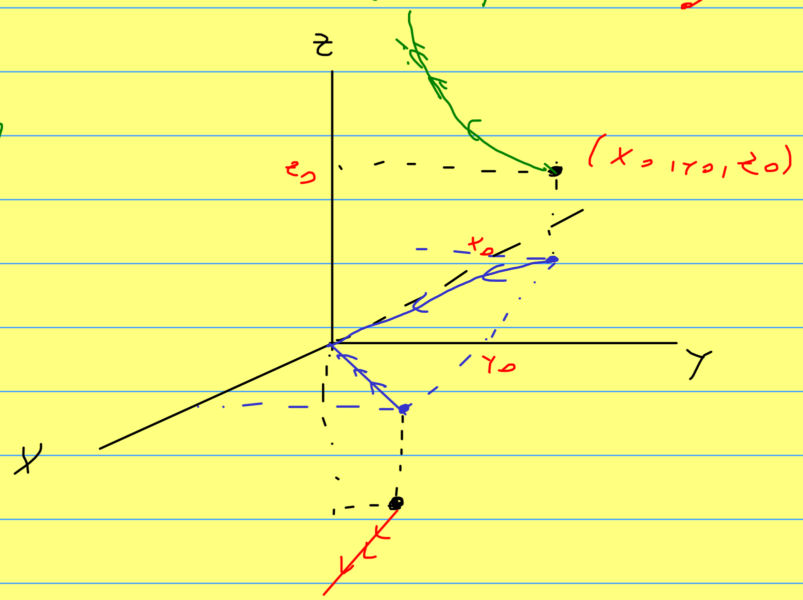
$y = \underbrace{\frac{B}{A}}_{m} x$

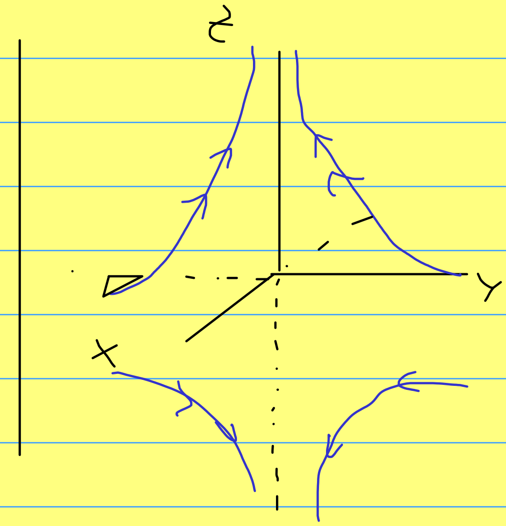
$X(t) = \begin{pmatrix} A e^{-t} \\ B e^{-t} \\ C e^{t} \end{pmatrix}$



Plano xy: $\begin{pmatrix} A e^{-t} \\ B e^{-t} \end{pmatrix}$

$\boxed{z(t) = C e^{t}}$





8. (a) $x'' + x' - 2x = 0$, $X' = AX$

$X(t) = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} \to X' = \begin{pmatrix} x' \\ x'' \end{pmatrix}$

$X' = \begin{pmatrix} x' \\ x'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \overset{0}{a}x + \overset{1}{b}x' \\ \underset{2}{c}x + \underset{-1}{d}x' \end{pmatrix}$

$X' = \begin{pmatrix} 0 & 1 \\ +2 & -1 \end{pmatrix} X$

Resolver esto y la segunda solución es la solución que buscan.