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e) $f'' + 2f' - 3f = \underbrace{\begin{cases} 1, & 0 \le t < c \\ 0, & t \ge c \end{cases}}_{g(t)}$ con condiciones iniciales $f(0) = f'(0) = 0$.

- $\mathcal{L}(f'' + 2f' - 3f) = \mathcal{L}(g)$

$\Rightarrow \mathcal{L}(f)[s^2 + 2s - 3] = \mathcal{L}(g)$

$$g(t) = \begin{cases} 1 & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

$$\mathcal{L}(g)(s) = \int_0^{+\infty} g(t) e^{-st}\,dt = \int_0^c 1 \cdot e^{-st}\,dt + \underbrace{\boxed{\int_c^{+\infty} 0\, e^{-st}\,dt}}_{=0}$$

$$\mathcal{L}(g)(s) = \int_0^c e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^c = \frac{1 - e^{-sc}}{s}$$

Esto vale si $0 \le t < c$

$$\boxed{\mathcal{L}(f)[s^2 + 2s - 3] = \frac{1 - e^{-sc}}{s}}$$

$$\mathcal{L}(f) = \frac{1 - e^{sc}}{(s^2 + 2s - 3)s}$$

- $s^2 + 2s - 3 = 0 \longrightarrow s = \frac{-2 \pm \sqrt{4 + 12}}{2} = \frac{-2 \pm \sqrt{16}}{2} = -1 \pm 2$ $\begin{cases} \to 1 \\ \to -3 \end{cases}$

$$\boxed{\mathcal{L}(f) = \frac{1 - e^{sc}}{(s-1)(s+3)s}}$$

$$\frac{1}{(s-2)(s+3)s} - \frac{e^{sc}}{(s-2)(s+3)s}$$

$$= \underbrace{\frac{1}{(s-2)(s+3)}}_{\mathcal{L}(h)} \cdot \underbrace{\frac{e^{sc}}{s}}_{\mathcal{L}(K)}$$

$$\mathcal{L}(h) = \frac{1}{(s-2)(s+3)} \qquad \mathcal{L}(K) = \frac{e^{sc}}{s}$$

$$\mathcal{L}(h)\,\mathcal{L}(K) = \mathcal{L}(h * K)$$

$$\mathcal{L}(h): \quad \frac{1}{(s-2)(s+3)} = \frac{A}{(s-2)} + \frac{B}{(s+3)}$$

$$A(s+3) + B(s-2) = 1 \longrightarrow \begin{cases} A + B = 0 & (i) \\ 3A - B = 1 & (ii) \end{cases}$$

$$(i) + (ii): \quad 4A = 1 \to \boxed{A = 1/4} \to B = -A \to \boxed{B = -1/4}$$

$$\mathcal{L}(h) = \frac{1}{4} \cdot \frac{1}{s-2} - \frac{1}{4} \cdot \frac{1}{s+3}$$

$$= \frac{1}{4}\left[\frac{1}{s-2} - \frac{1}{s+3}\right]$$

$$\Rightarrow \boxed{h(t) = \frac{e^{t} - e^{-3t}}{4}}$$

e^{-as} donde $-a = c$

$$\mathcal{L}(K) = \frac{e^{sc}}{s} = \overbrace{\frac{1}{s}}^{F(s)} \cdot e^{sc}$$

$1 = H(t)$

b) Demostrar que si $F(s)$ es la transformada de Laplace de $f(t)$ entonces $F(s)e^{-as}$ es la transformada de $f(t-a)$, donde a es un número real fijo , y $f(t) = 0$ para todo real $t < 0$ y también para todo $t < a$. Este resultado se llama "propiedad de traslación en el tiempo".

$$\mathcal{L}(K) = \frac{e^{sc}}{s} = \mathcal{L}(H(t+c)) \to K(t) = H(t+c) = \begin{cases} 1 & \text{si } t \geq -c \\ 0 & \text{si } t < -c \end{cases}$$

$$\mathcal{L}(h) = \frac{1}{(s-2)(s+3)} \to h(t) = \frac{e^{t} - e^{-3t}}{4}$$

$$\mathcal{L}(K) = \frac{e^{sc}}{s} \to K(t) = H(t+c)$$

$$\mathcal{L}(h)\mathcal{L}(K) = \mathcal{L}(h * K)$$

$$\mathcal{L}(s) = \frac{1}{(s-2)(s+3)s} - \mathcal{L}(h * K)$$

$$\mathcal{L}(Ae^{t} + Be^{-3t} + C)$$

$$\boxed{f(t) = Ae^{t} + Be^{-3t} + C - h(t) * K(t)}$$

(vii) $e^{\alpha t}t$ (viii) $e^{\alpha t}t^2$ (ix) $e^{\alpha t}t^n$.

$\mathcal{L}(e^{\alpha t}t)$

f) Admitiendo que la transformada de Laplace $F(s)$ de $f(t)$ es derivable como función de variable s, y que al derivar respecto a s la integral impropia que define $F(s)$ es igual a la integral impropia de la función que se obtiene derivando dentro de la integral, deducir que:

$$\frac{dF}{ds} = F'(s) = (\mathcal{L}[-tf(t)])(s)$$

$$te^{\alpha t} = -tf(t) \rightarrow f(t) = -e^{\alpha t}$$

$$\mathcal{L}(f) = F(s) = -\mathcal{L}(e^{\alpha t}) = -\frac{1}{s-\alpha} \rightarrow \boxed{F(s) = -\frac{1}{s-\alpha}}$$

$$F'(s) = \frac{(\cancel{-1})'(s-\alpha) + 1\,(s-\alpha)'}{(s-\alpha)^2} = \frac{1}{(s-\alpha)^2}$$

$$\boxed{\mathcal{L}(te^{\alpha t}) = \frac{1}{(s-\alpha)^2}}$$

$$f(t) = -te^{\alpha t}$$

• $\mathcal{L}(t^2e^{\alpha t})$, $t^2e^{\alpha t} = -t\,f(t)$

$$F(s) = \mathcal{L}(-te^{\alpha t}) = -\mathcal{L}(te^{\alpha t}) = -\frac{1}{(s-\alpha)^2}$$

$$\mathcal{L}(t^2e^{\alpha t}) = F'(s) = \frac{(\cancel{-1})'(s-\alpha)^2 + 1\,((s-\alpha)^2)'}{(s-\alpha)^4} \quad ((s-\alpha)^2)' \rightarrow 2(s-\alpha)$$

$$= \frac{2(s-\alpha)}{(s-\alpha)^4} = \frac{2}{(s-\alpha)^3}$$

$$\boxed{\mathcal{L}(t^2 e^{\alpha t}) = \frac{2}{(s-\alpha)^3}}$$

- $\mathcal{L}(t^n e^{\alpha t}) = \dfrac{n!}{(s-\alpha)^{n+1}}$

- Caso base: Con $n=1$, o $n=2$ vimos que funciona.

- Caso inductivo: Supongamos $\mathcal{L}(t^{n-1} e^{\alpha t}) = \dfrac{(n-1)!}{(s-\alpha)^{(n-1)+1}}$

Queremos probar $\mathcal{L}(t^n e^{\alpha t}) = \dfrac{n!}{(s-\alpha)^{n+1}}$

Sabemos: $F(s) = \mathcal{L}(f(t)) \Rightarrow \mathcal{L}(-t f(t)) = F'(s)$

$$t^n e^{\alpha t} = -t\left(-t^{n-1} e^{\alpha t}\right)$$

$$\mathcal{L}(t^n e^{\alpha t}) = \mathcal{L}\Big(-t\,\big(\overbrace{-t^{n-1} e^{\alpha t}}^{f(t)}\big)\Big) = \Big(\mathcal{L}(-t^{n-1} e^{\alpha t})\Big)'$$

Por hipótesis inductiva $\mathcal{L}(-t^{n-1} e^{\alpha t}) = -\mathcal{L}(t^{n-1} e^{\alpha t})$

$$= -\frac{(n-1)!}{(s-\alpha)^n}$$

$$\left(-\frac{(n-1)!}{(s-\alpha)^n}\right)' = \frac{(-(n-1)!)'(s-\alpha)^n + (n-1)!\,\left((s-\alpha)^n\right)'}{(s-\alpha)^{2n}}$$

$(-(n-1)!)' = 0$

$\left((s-\alpha)^n\right)' = n(s-\alpha)^{n-1}$

$n-1-2n = -n-1 = -(n+1)$

$$= \frac{(n-1)!\,n\,(s-\alpha)^{n-1}}{(s-\alpha)^{2n}}$$

$(n-1)!\,n = n!$

$$= n!\,(s-\alpha)^{-(n+1)} = \frac{n!}{(s-\alpha)^{n+1}}$$

$$\boxed{\mathcal{L}(t^n e^{\alpha t}) = \frac{n!}{(s-\alpha)^{n+1}}}$$

(x) $t^n \operatorname{sen}(\alpha t)$

$$\mathcal{L}(t\,\operatorname{sen}(\alpha t)) = (-\mathcal{L}(\operatorname{sen}(\alpha t)))'$$

$$\operatorname{sen}(\alpha t) = \frac{e^{i\alpha t} - e^{-i\alpha t}}{2i}$$

$$\mathcal{L}(\operatorname{sen}(\alpha t)) = \frac{1}{2i}\left(\mathcal{L}(e^{i\alpha t}) - \mathcal{L}(e^{-i\alpha t})\right)$$

$$= \frac{1}{2i}\left[\frac{1}{s-i\alpha} - \frac{1}{s+i\alpha}\right]$$

$-\frac{i}{2}$

$\frac{1}{i} = -i$

$$= -\frac{i}{2}\left[\frac{\cancel{s} + i\alpha - \cancel{s} + i\alpha}{s^2 - (i\alpha)^2}\right] = -\frac{i^2\alpha}{s^2-(i\alpha)^2} = \frac{\alpha}{s^2+\alpha^2}$$

$2i\alpha$, -1

$$(i\alpha)^2 = i^2\alpha^2 = -\alpha^2$$

$$\boxed{\mathcal{L}(\operatorname{sen}(\alpha t)) = \frac{\alpha}{s^2+\alpha^2}}$$

$$\mathcal{L}(t\,\operatorname{sen}(\alpha t)) = (-\mathcal{L}(\operatorname{sen}(\alpha t)))' = \left(-\frac{\alpha}{s^2+\alpha^2}\right)' = \frac{2s\alpha}{(s^2+\alpha^2)^2}$$

$$x' + f(t)x = 0$$

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

<u>Notación</u>: $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $X' = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$X' = AX$$