17/11

8. Estudiar la convergencia puntual y uniforme (calculando la suma) para la serie de funciones $\sum_{n=1}^{\infty} a_n(x)$ con $a_n(x) = \left(\frac{1-x}{1+x}\right)^n$.

$$S_K(x) = \sum_{n=1}^{K} a_n(x) = \sum_{n=1}^{K} \left(\frac{1-x}{1+x}\right)^n$$

• $S_K(x) \xrightarrow{c.p.} f(x)$ si:

$\forall x_0$ fijo: $\lim_{K\to+\infty} |S_K(x_0) - f(x_0)| = 0$

$$|S_K(x_0) - S_\infty(x_0)| = \sum_{n=K+1}^{\infty} a_n(x_0) \xrightarrow[K]{} 0$$

Obs: Si $x=0 \Rightarrow a_n(0) = 1 \Rightarrow \sum_{n=K+1}^{\infty} 1 \not\xrightarrow[K]{} 0$

$$a_n(x_0) = a_n = \left(\frac{1-x_0}{1+x_0}\right)^n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} a_n$$

• Si $\sum_{n=1}^{\infty} a_n$ converge $\Rightarrow \sum_{n=K+1}^{+\infty} a_n \xrightarrow[K]{} 0$

. Raíz enésima: $\lim_n \sqrt[n]{|a_n|} = C$

Si $C < 1 \Rightarrow$ La serie converge

$C > 1 \Rightarrow$ Diverge

$C = 1$ no sabemos

$$a_n = \left(\frac{1-x_0}{1+x_0}\right)^n \quad \Rightarrow \sqrt[n]{|a_n|} = \left|\frac{1-x_0}{1+x_0}\right|$$

$$\lim_n \sqrt[n]{|a_n|} = \left|\frac{1-x_0}{1+x_0}\right|$$

→ Hay que estudiar cuando este cociente es menor a 1.

- $\underline{x_0 > 0}$: $\left|\frac{1-x_0}{1+x_0}\right| < 1 \longrightarrow (0,+\infty)$

- $x_0 = 0$: Diverge

- $\underline{x_0 < 0}$: $1 - x_0 = 1 + |x_0|$

$|1+x_0| \leq |x_0|$

$$\left|\frac{1-x_0}{1+x_0}\right| = \frac{1+|x_0|}{|1+x_0|} \geq \frac{1+|x_0|}{|x_0|} = \overset{\to 0}{\frac{1}{|x_0|}} + \overset{1}{\frac{|x_0|}{|x_0|}}$$

> 1

$$S_K(x) \xrightarrow{C.P.} S_\infty(x) \quad \text{para } x \in (0,+\infty)$$

- $\lim\limits_{K} \sup\limits_{x\in(0,+\infty)} |S_K(x) - S_\infty(x)| = 0$

- $\underset{x\in(0,+\infty)}{a_n(x)} \leq A_n$ y $\sum A_n$ converge

entonces $S_K(x) \xrightarrow{C.U.} S_\infty(x)$

$$a_n(x) = \left(\frac{1-x}{1+x}\right)^n \qquad x \in (0,+\infty)$$

$A_n \in \mathbb{R}$ $\forall$ $\underset{x\in(0,+\infty)}{a_n(x) \leq A_n}$ y además $\sum_{n=1}^{+\infty} A_n$ sea convergente

$1-x \leq 1 \rightarrow a_n(x) = \left(\frac{1-x}{1+x}\right)^n \leq \left(\frac{1}{1+x}\right)^n$

$x>0$

$$a_n(x) \leq \left(\frac{1}{1+x}\right)^n$$

$$\sum_{n=1}^{\infty} a_n(x) \leq \sum_{n=1}^{+\infty} \left(\frac{1}{1+x}\right)^n$$

• Si $r>0$: $\sup_{x\in(r,+\infty)} \left(\frac{1}{1+x}\right) = \frac{1}{1+r} < 1$ $\quad r>0$

$$\lim_{K\to+\infty} \sup_{x\in(r,+\infty)} \left| S_K(x) - S_\infty(x) \right| \leq \sup_{x\in(r,+\infty)} \left| \sum_{n=K+1}^{+\infty} \left(\frac{1}{1+x}\right)^n \right| = 0$$

Obs: $\sum_{n=1}^{+\infty} \ell^n$ converge si $|\ell|<1$ → Serie geométrica

$\Rightarrow$ Si $|\ell|<1$, $\sum_{n=K+1}^{+\infty} \ell^n \xrightarrow[K]{} 0$

• Si $r=0$: $\sup_{x\in(0,+\infty)} \frac{1}{1+x} = \frac{1}{1+0} = 1$

Luego $\sum_{n=K+1}^{+\infty} 1$ diverge

$$\Rightarrow \lim_{K} \sup_{x\in(0,+\infty)} \left|S_K(x) - S_\infty(x)\right|$$

$$\parallel$$

$$\lim_{K} \sup_{x\in(0,+\infty)} \left|\sum_{n=K+1}^{+\infty} \left(\frac{1}{1+x}\right)^n\right|$$

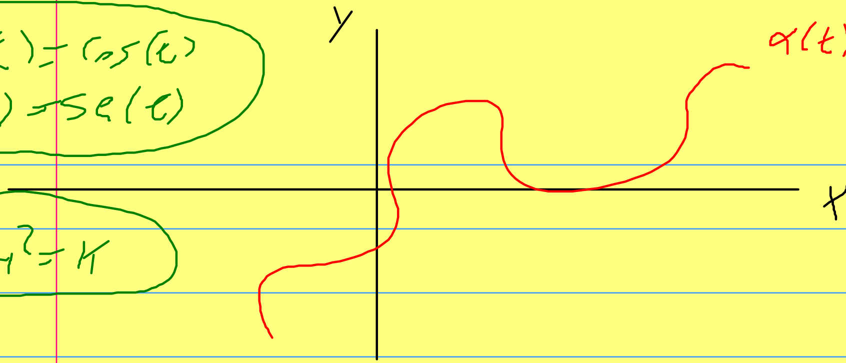
$$\parallel$$

$$\lim_{K} \left|\sum_{n=K+1}^{\infty} 1\right| \neq 0 \quad \rightarrow \text{ por sup } \sum_{n=K+1}^{\infty} 1 = \infty \quad \forall K$$

c.p. en $(0,+\infty)$, pero c.u. en $(r,+\infty)$ $\forall\, r>0$

$x(t) = \cos(t)$
$y(t) = \mathrm{sen}(t)$

$x^2 + y^2 = K$



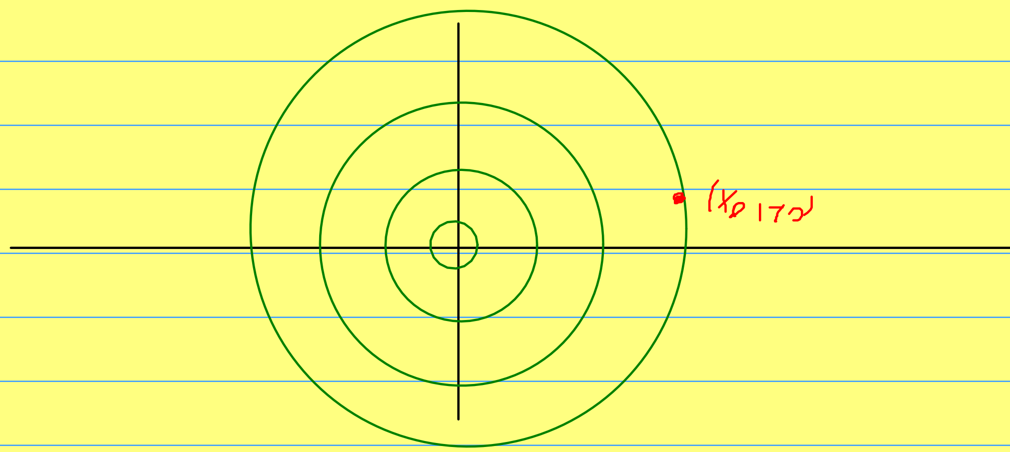
$\alpha(t) = (x(t), y(t))$

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 = -1 \Rightarrow \lambda = \pm i \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = P \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} P^{-1}$$

$$e^{a+ib} = e^{a}(\cos b + i \,\mathrm{sen}\, b)$$



3. **Parcial 2008.** Sea la ecuación

$$\begin{cases} u_t = u_{xx} & (t,x) \in (0,+\infty)\times(0,\pi) \\ u(t,0) = 0 \text{ y } u_x(t,\pi) = 0 & t > 0 \\ u(0,x) = \underline{x(2\pi - x)} & x \in [0,\pi] \end{cases}$$

$\longrightarrow 3$

$2L$

a) Si $u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$ es solución del problema entonces

$u_k(x,t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$u(x,0) = \sum_{K=1}^{\infty} u_K(x,0) = x(2\pi - x)$$

$x \in (0,\pi)$:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_K = \frac{2}{\pi}\int_0^{\pi} f(x)\cos Kx\,dx$$

$$b_K = \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen} Kx\,dx$$

$$u(x,t) = \sum_{K=1}^{+\infty} u_K(x,t)$$

$$\sum_{K=1}^{+\infty} \frac{\partial}{\partial t} u_K(x,t) = \sum_{K=1}^{+\infty} \frac{\partial^2}{\partial x^2} u_K(x,t)$$

$$\sum_{K=1}^{+\infty} \frac{\partial}{\partial x} u_K(\pi,t) = 0 \longrightarrow \frac{\partial}{\partial x} u_K(t,\pi) = 0 \quad \forall K$$

$\to X'(\pi) = 0$

$$\sum_{K=1}^{+\infty} u_K(0,t) = 0 \longrightarrow \underline{u_K(0,t)} = 0$$

$\to X(0) = 0$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \rightarrow X T' = X'' T$$

$$\frac{T'}{T}(t) = \frac{X''}{X}(x) = K$$

$$\left\{ T' - K T = 0 \rightarrow T' = K T \rightarrow T(t) = A e^{Kt} \right.$$

$$\left\{ \begin{array}{l} X'' - K X = 0 \\ X(0) = 0 \\ X'(\pi) = 0 \end{array} \right.$$

• $K=0$: $\left.\begin{array}{l} X(x) = Ax + B \\ X(0) = 0 \rightarrow B = 0 \\ X'(\pi) = 0 \rightarrow A = 0 \end{array}\right\} \cancel{X(x) = 0}$

• $K>0$: $\lambda = \pm\sqrt{K} \rightarrow X(x) = A e^{\sqrt{K}x} + B e^{-\sqrt{K}x}$

$$0 = X(0) = A + B \rightarrow B = -A \qquad B = 0 \qquad \boxed{X(x) = 0}$$

$$0 = X'(\pi) = \sqrt{K}\,A\,[\underbrace{e^{\sqrt{K}\pi} + e^{-\sqrt{K}\pi}}_{\neq 0}] \rightarrow A = 0$$

• $K<0$: $X(x) = A\cos(\sqrt{|K|}\,x) + B\sin(\sqrt{|K|}\,x)$

$$0 = X(0) = A \rightarrow A = 0$$

$$0 = X'(\pi) = \sqrt{|K|}\,B\cos(\sqrt{|K|}\,\pi) \rightarrow \sqrt{|K|}\,\pi = \frac{\pi}{2} + n\pi$$

$$\sqrt{|k|} = -\frac{\pi}{2} + n\pi$$

$$|k| = \left(-\frac{\pi}{2} + n\pi\right)^2$$

$$k = -\left(-\frac{\pi}{2} + n\pi\right)^2$$

$$\boxed{X_n(x) = B_n \operatorname{sen}\left(\left(-\frac{\pi}{2} + n\pi\right) x\right)}$$

$$\boxed{T_n(t) = A_n\, e^{-\left(-\frac{\pi}{2} + n\pi\right)^2 t}}$$

$$u_n(x,t) = C_n \operatorname{sen}\left(\left(-\frac{\pi}{2} + n\pi\right) x\right) e^{-\left(-\frac{\pi}{2} + n\pi\right)^2 t}$$

$$x(2\pi - x) = u(x,0) = \sum_{n=1}^{+\infty} u_n(x,0) = \sum_{n=1}^{+\infty} \boxed{C_n} \operatorname{sen}\left(\left(\underbrace{-\frac{\pi}{2} + n\pi}_{\frac{n\pi}{L}}\right) x\right)$$

$\Rightarrow$ C_n tienen que ser los coeficientes de alguna extensión impar de $x(2\pi - x)$

$$\cdot\ u_t = u_{xx} \quad \to \quad \frac{\partial}{\partial t} \sum u_n = \sum \frac{\partial}{\partial t} u_n$$

$$\downarrow$$

$$\sum_{n=1}^{K} \frac{\partial}{\partial t} u_n(x,t) \xrightarrow{\text{c.u.}} h(x,t)$$