15/11

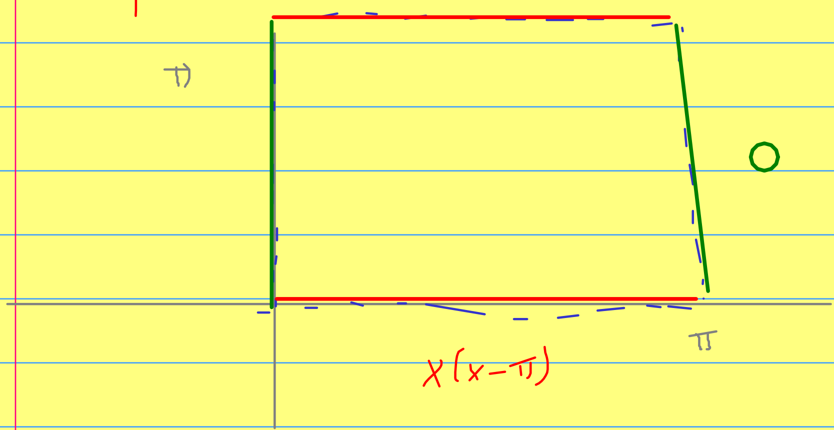
7. Utilizar el método de las series de Fourier para resolver la ecuación

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \qquad (x,y) \in (0,\pi) \times (0,\pi)$$

con las condiciones de borde:

a) $u(0,y) = u(\pi,y) = \sin y$, $u(x,\pi) = u(x,0) = 0$,
b) $u(0,y) = u(\pi,y) = 0$, $u(x,\pi) = u(x,0) = x(x-\pi)$,
c) $u(0,y) = u(\pi,y) = \sin y$, $u(x,\pi) = u(x,0) = x(x-\pi)$.

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in (0,\pi)^2 \\ u(0,y) = u(\pi,y) = 0 \\ u(x,\pi) = u(x,0) = x(x-\pi) \end{cases}$$



Suponer que $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$u_{xx} + u_{yy} = 0 \Rightarrow X''Y + XY'' = 0$

$$\frac{X''}{X}(x) = -\frac{Y''}{Y}(y) = K$$

$u(0,y) = u(\pi,y) = 0 \Rightarrow \boxed{X(0) = X(\pi) = 0}$

$u(x,0) = u(x,\pi) = x(x-\pi) \Rightarrow \boxed{\begin{array}{l} X(x) = x(x-\pi) \\ Y(0) = Y(\pi) = 1 \end{array}}$

$$\begin{cases} X'' - KX = 0 \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} \qquad \begin{cases} Y'' - KY = 0 \\ Y(0) = 1 \\ Y(\pi) = 1 \end{cases}$$

$\underline{K=0}$:

$$\left.\begin{aligned} &X(x) = Ax + B \\ &X(0) = B \to B = 0 \\ &X(\pi) = A\pi \to A = 0 \end{aligned}\right\} \boxed{X(x) = 0}$$

$\underline{K>0}$: $\lambda^2 = K$, $\lambda = \pm\sqrt{K} \in \mathbb{R}$

$$X(x) = A e^{\sqrt{K}x} + B e^{-\sqrt{K}x}$$

$$\left.\begin{aligned} &0 = X(0) = A + B \to B = -A \\ &0 = X(\pi) = A\left(e^{\sqrt{K}\pi} - e^{-\sqrt{K}\pi}\right) \longrightarrow A = 0 \end{aligned}\right\} X(x) = 0$$

$\underline{K<0}$: $\lambda = \pm\sqrt{|K|}\, i$

$$X(x) = A\cos(\sqrt{|K|}x) + B\sin(\sqrt{|K|}x)$$

$$0 = X(0) = A \to A = 0$$

$$0 = X(\pi) = \underbrace{B}_{B\neq 0}\sin(\sqrt{|K|}\pi) \to \sin(\sqrt{|K|}\pi)$$

$$\to \sqrt{|K|}\pi = n\pi$$

$$\to |K| = n^2 \to \boxed{K = -n^2}$$

$$X(x) = B\sin(nx) = B\sin(nx)$$

$T(0)\, x(x) = x(\pi - x)$

Extension impar $\to a_k = 0$

$L = \pi$

$x \in (0, \pi)$

$$f(x) = x(\pi - x)$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\, \operatorname{sen}(kx)\, dx$$

$$= \frac{2}{\pi}\left(\frac{\pi\overbrace{\cancel{\sin(\pi k)}}^{0} - \pi \cancel{k}\overbrace{\cos(\pi k)}^{(-1)^k}}{k^{\cancel{2}}} - \frac{-\pi^2 \cancel{k^2}\overbrace{\cos(\pi k)}^{(-1)^k} + 2\left(\overbrace{\cancel{\pi k \sin(\pi k)}}^{0} + \overbrace{\cos(\pi k)}^{(-1)^k}\right) - 2}{k^3} \right)$$

$$b_k = \frac{2}{\pi}\left[\frac{-\pi(-1)^k}{k} + \frac{\pi^2(-1)^k}{k} - \frac{2(-1)^k}{k^3} + \frac{2}{k^3} \right]$$

$$= \frac{2}{k}\underbrace{\left(\pi(-1)^k - (-1)^k \right)}_{0} + \frac{4}{\pi k^3}\left[1 - (-1)^k \right]$$

$+(-1)^{k+1}$

$$b_k = \frac{4}{\pi k^3}\left(1 - (-1)^k \right) \to 0 \quad k \text{ par}$$

$$x(\pi - x) = \sum_{k \text{ impar}} \frac{8}{\pi k^3} \operatorname{sen}(kx) = \frac{8}{\pi} \sum_{k \text{ impar}} \frac{1}{k^3} \operatorname{sen}(kx)$$

$x \in (0, \pi)$

$$x(x) = \frac{8}{\pi} \sum_{k \text{ impar}} \frac{\operatorname{sen}(kx)}{k^3}$$

$$\begin{cases} Y'' + \alpha Y = 0 \\ Y(0) = 1 \\ Y(\pi) = 2 \end{cases} \qquad \alpha = -n^2$$

$$\lambda^2 = n^2 \to \lambda = \pm n$$

$$Y(y) = A e^{ny} + B e^{-ny}$$

$$\left.\begin{aligned} 1 = Y(0) &= A + B \\ 2 = Y(\pi) &= A e^{n\pi} + B e^{-n\pi} \end{aligned}\right\} \begin{cases} A + B = 1 \\ A e^{n\pi} + B e^{-n\pi} = 2 \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ e^{n\pi} & e^{-n\pi} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\hookrightarrow \det = e^{-n\pi} - e^{n\pi} \neq 0 \quad \to \quad S.C.D.$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{e^{-n\pi} - e^{n\pi}} \begin{pmatrix} e^{-n\pi} & -1 \\ -e^{n\pi} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{e^{-n\pi} - e^{n\pi}} \begin{pmatrix} e^{-n\pi} - 2 \\ -e^{n\pi} + 2 \end{pmatrix} \qquad A = \frac{e^{-n\pi} + 2}{e^{-n\pi} - e^{n\pi}} \qquad B = \frac{-e^{n\pi} + 2}{e^{-n\pi} - e^{n\pi}}$$

$$X_n(x) = A_n \operatorname{sen}(nx)$$

$$Y_n(y) = \underbrace{\frac{e^{-n\pi} + 2}{e^{-n\pi} - e^{n\pi}}}_{A} e^{ny} + \underbrace{\frac{2 - e^{n\pi}}{e^{-n\pi} - e^{n\pi}}}_{B} e^{-ny}$$

$$u_n(x,y) = X_n(x) Y_n(y) \Rightarrow u(x,y) = \sum_{n=1}^{+\infty} u_n(x,y)$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{e^{-n\pi}+1}{e^{-n\pi}-e^{n\pi}}\right) e^{ny} + \frac{1-e^{n\pi}}{e^{-n\pi}-e^{n\pi}}\, e^{-ny}\right] \sin(nx)$$

$$x(x-\pi) = u(x,0) = \sum_{n=1}^{\infty} \underbrace{A_n\, \frac{2+e^{-n\pi}-e^{n\pi}}{e^{-n\pi}-e^{n\pi}}}_{b_n} \sin(nx)$$

$$A_n \left(\frac{2+e^{-n\pi}-e^{n\pi}}{e^{-n\pi}-e^{n\pi}}\right) = \frac{4}{\pi n^3}\left(1-(-1)^n\right)$$

$$A_n = \frac{4}{\pi n^3}\left(1-(-1)^n\right)\left(\frac{e^{-n\pi}-e^{n\pi}}{2+e^{-n\pi}-e^{n\pi}}\right)$$

$$u(x,y) = \sum_{n=1}^{+\infty} \frac{4}{\pi n^3}\left(1-(-1)^n\right)\left(\frac{e^{-n\pi}-e^{n\pi}}{2+e^{-n\pi}-e^{n\pi}}\right)\left[\left(\frac{e^{-n\pi}+1}{e^{-n\pi}-e^{n\pi}}\right)e^{ny} + \left(\frac{1-e^{n\pi}}{e^{-n\pi}-e^{n\pi}}\right)e^{-ny}\right]\sin(nx)$$

En resumen:

$\cdot\, u(x,y) = X(x)\,Y(y)$.

1) Aplicar variables separables

$X_n(x)\,,\ Y_n(y) \Rightarrow u_n(x,y) = \underbrace{X_n(x)\,Y_n(y)}$

Van a quedar constantes a determinar

2) Viendo que $u(x,y) = \sum_{n=1}^{+\infty} u_n(x,y)$

investigar si es una serie de Fourier, de cosenos o de senos

3) Hallando según corresponda la serie de Fourier, de cosenos o de senos, e igualando coeficientes podemos despejar las constantes de la parte 1