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3. **Parcial 2008.** Sea la ecuación $\rightarrow$ Ecuación de calor.

$$\begin{cases} \boxed{u_t = u_{xx}} & (t,x) \in (0,+\infty)\times(0,\pi) \\ u(t,0) = 0 \text{ y } u_x(t,\pi) = 0 & t > 0 \\ \boxed{u(0,x) = x(2\pi - x)} & x \in [0,\pi] \end{cases}$$

$u_0(x) \rightarrow$

a) Si $u(x,t) = \sum_{k=1}^{\infty} \boxed{u_k(x,t)}$ es solución del problema entonces

$u_k(x,t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Teorema 0.1.

Sea $\boxed{u_0(x) = \sum_{k=1}^{\infty} b_k sen\left(\frac{k\pi}{L}x\right)}$ *condición inicial del problema de Cauchy-Dirichlet . Si* $\sum_{k=1}^{+\infty} |b_k|$ *es convergente entonces:*

$$U(t,x) = \sum_{k=1}^{+\infty} \boxed{b_k sen\left(\frac{k\pi}{L}x\right) e^{-\left(\frac{k\pi}{L}\right)^2 t}} \qquad (0.8)$$

$u_0(x) = x(2\pi - x) \qquad x \in (0,\pi) \rightarrow \boxed{L = \pi}$

Extensión impar: $a_k = 0$

$$b_k = \frac{2}{\pi}\int_0^{\pi} u_0(x)\, sen(kx)\, dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} x(2\pi - x)\, sen(kx)\, dx$$

$$= \frac{2}{\pi}\left[\underbrace{2\pi\int_0^{\pi} x\, sen\, kx\, dx}_{\mathrm{I}} - \underbrace{\int_0^{\pi} x^2\, sen\, kx\, dx}_{\mathrm{II}}\right]$$

Partes

$$\mathrm{I} \quad \int_0^{\pi} x\, sen\, kx\, dx = \underbrace{-\frac{x\cos kx}{k}\Big|_0^{\pi}}_{-\frac{\pi}{k}(-1)^k} + \frac{1}{k}\int_0^{\pi} \cos kx\, dx$$

Primitiva: $\frac{sen\, kx}{k}$

$$\int_0^\pi x \operatorname{sen} Kx\,dx = -\frac{\pi}{K}(-1)^K + \underbrace{\frac{1}{K^2}\operatorname{sen} Kx\Big|_0^\pi}_{0}$$

$$\text{I} = -\frac{\pi}{K}(-1)^K$$

$$\text{II} \quad \int_0^\pi x^2 \operatorname{sen} Kx\,dx \overset{\text{partes}}{=} \underbrace{-\frac{x^2 \cos Kx}{K}\Big|_0^\pi}_{-\frac{\pi^2}{K}(-1)^K} + \frac{2}{K}\int_0^\pi x \cos Kx\,dx$$

$$\int_0^\pi x^2 \operatorname{sen} Kx\,dx = -\frac{\pi^2}{K}(-1)^K + \frac{2}{K}\left[\underbrace{\cancel{\frac{x \operatorname{sen} Kx}{K}}\Big|_0^\pi}_{0} - \frac{1}{K}\int_0^\pi \operatorname{sen} Kx\,dx\right]$$

primitiva $-\dfrac{\cos Kx}{K}$

$$\int_0^\pi x^2 \operatorname{sen} Kx\,dx = -\frac{\pi^2}{K}(-1)^K + \overbrace{\frac{2}{K}\left[-\frac{1}{K}\left(-\frac{\cos Kx}{K}\right.\right.}^{\frac{2}{K^3}}\left.\left.\Big|_0^\pi\right)\right]$$

$$= -\frac{\pi^2}{K}(-1)^K + \frac{2}{K^3}\left[(-1)^K - 1\right]$$

$$\text{II} = -\frac{\pi^2}{K}(-1)^K + \frac{2}{K^3}\left[(-1)^K - 1\right]$$

$$b_K = \frac{2}{\pi}\left[2\pi\overbrace{\left(-\frac{\pi}{K}(-1)^K\right)}^{\text{I}} - \overbrace{\left(-\frac{\pi^2}{K}(-1)^K + \frac{2}{K^3}\left[(-1)^K - 1\right]\right)}^{\text{II}}\right]$$

$$b_K = \frac{2}{\pi}\left[\frac{\pi}{K}(-1)^K\left[\overbrace{-2\pi+\pi}^{-\pi}\right] - \frac{2}{K^3}\left((-1)^K - 1\right)\right)$$

$$b_K = \frac{2}{\pi}\left[-\frac{\pi^2}{K}(-1)^K - \frac{2}{K^3}\left((-1^K)-1\right)\right]$$

No converge

$$\frac{\pi}{2}|b_K| \leq \frac{\pi^2}{K} + \frac{2}{K^3}\left(1-(-1)^K\right)$$

Ecuación de calor.

3. **Parcial 2008.** Sea la ecuación

$$\begin{cases} \boxed{u_t = u_{xx}} & (t,x) \in (0,+\infty)\times(0,\pi) \\ u(t,0) = 0 \text{ y } u_x(t,\pi) = 0 & t > 0 \\ \boxed{u(0,x) = x(2\pi - x)} & x \in [0,\pi] \end{cases}$$

$u_0(x)$

a) Si $u(x,t) = \sum_{k=1}^{\infty} u_k(x,t)$ es solución del problema entonces

$u_k(x,t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

- $u(t,x) = T(t)\,\mathcal{X}(x)$

$$T'\mathcal{X} = T\mathcal{X}'' \Rightarrow \frac{\mathcal{X}''}{\mathcal{X}}(x) = \frac{T'}{T}(t) = K$$

$$\begin{cases} \mathcal{X}'' - K\mathcal{X} = 0 \\ T' - KT = 0 \end{cases} \rightarrow T' = KT \Rightarrow \boxed{T(t) = Ae^{Kt}}$$

$$K = -\left(\frac{1}{2}+n\right)^2$$

$$\lambda^2 - K = 0 \quad :$$

$$\begin{cases} u_t = u_{xx} & (t,x) \in (0,+\infty) \times (0,\pi) \\ u(t,0) = 0 \text{ y } u_x(t,\pi) = 0 & t > 0 \\ u(0,x) = x(2\pi - x) & x \in [0,\pi] \end{cases}$$

$0 = u(t,0) = T(t)X(0) \rightarrow \boxed{X(0) = 0}$

$0 = u_x(t,\pi) = T(t)X'(\pi) \rightarrow \boxed{X'(\pi) = 0}$

$$\begin{cases} X'' - KX = 0 \\ X(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

$\lambda^2 = K$

$\underline{K=0}$: $X''(x) = 0 \rightarrow X(x) = ax + b$

$X(0) = 0 \rightarrow \boxed{b = 0}$

$X'(\pi) = a = 0$ sii $\boxed{a = 0}$

Función trivial

$\underline{K>0}$: $\lambda = \pm\sqrt{K} \in \mathbb{R}$

$X(x) = Be^{\sqrt{K}x} + Ce^{-\sqrt{K}x}$

$0 = X(0) = B + C \rightarrow \boxed{C = -B}$

$X'(x) = \sqrt{K}\left[Be^{\sqrt{K}x} - Ce^{-\sqrt{K}x}\right]$ B

$0 = X'(\pi) = \underline{\sqrt{K}}\,B\left[\underline{e^{\sqrt{K}\pi}} + \underline{e^{-\sqrt{K}\pi}}\right]$

$\boxed{C = 0}$

$\boxed{B = 0}$

$\underline{K<0}$: $\lambda^2 = K \Rightarrow \lambda = 0 \pm \sqrt{|K|}\, i$

$$X(x) = B \cos(\sqrt{|K|}\,x) + C \operatorname{sen}(\sqrt{|K|}\,x)$$

$\to 0$

$$0 = X(0) = B \to \boxed{B=0}$$

$$X'(x) = \sqrt{|K|}\, C \cos(\sqrt{K}\, x)$$

$$0 = X'(\pi) = \sqrt{|K|}\, C \cos(\sqrt{|K|}\,\pi) \to \cos(\sqrt{|K|}\,\pi) = 0$$

$\sqrt{|K|} \vee 0$, $C \neq 0$

$$\sqrt{|K|}\,\pi = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$$

$$\sqrt{|K|} = \frac{1}{2} + n$$

$$K = -\left(\frac{1}{2}+n\right)^2$$

$$\boxed{X(x) = C \operatorname{sen}\left(\left(\frac{1}{2}+n\right)x\right)}$$

$$\begin{cases} u_t = u_{xx} & (t,x) \in (0,+\infty)\times(0,\pi) \\ u(t,0) = 0 \text{ y } u_x(t,\pi) = 0 & t>0 \\ u(0,x) = \underline{x(2\pi - x)} & x \in [0,\pi] \end{cases}$$

D

$$u(t,x) = C \operatorname{sen}\left(\left(\frac{1}{2}+n\right)x\right) A e^{-\left(\frac{1}{2}+n\right)^2 t}$$

$$u(t,x) = D \operatorname{sen}\left(\left(\frac{1}{2}+n\right)x\right) e^{-\left(\frac{1}{2}+n\right)^2 t}$$

$$u(0,x) = \underline{D \operatorname{sen}\left(\left(\frac{1}{2}+n\right)^{\pi} x\right)}$$

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)$$

$$\begin{cases} u_t = u_{xx} & (t,x)\in(0,+\infty)\times(0,\pi) \\ u(t,0)=0 \text{ y } u_x(t,\pi)=0 & t>0 \\ \boxed{u(0,x)=x(2\pi-x)} & x\in[0,\pi] \end{cases}$$

$$0=u(t,0)=\sum_{k=1}^{\infty}u_k(0,t)$$

$u_k(0,t)=0 \quad \forall t$

$u_k(x,t)=a\underbrace{x^b}_{h(x)}f(t)$

Terminos positivos y negativos

Si $\sum u_k(x,t) \rightrightarrows u(x,t) \Rightarrow u_x(x,t)=\sum \frac{\partial u_k}{\partial x}(x,t)$

$$\sum \frac{\partial u_k}{\partial x}(x,t) \rightrightarrows \frac{\partial u}{\partial x}(x,t)$$

$$0=u_x(\pi,t)=\sum_{k=1}^{\infty}\frac{\partial u_k}{\partial x}(\pi,t)$$

$\frac{\partial u_k}{\partial x}(\pi,t)=0 \quad \forall t$

$\frac{\partial u_k}{\partial x}=c\underbrace{(x-\pi)^{\ell}}_{\bar{g}(x)}g(t)$

$\ell>1$

$$u_k(x,t)=A\,x^n(x-\pi)^{\ell}\,g(t)$$

$$u_k(0,t)=0$$

$$\frac{\partial u_k}{\partial x}=nAx^{n-1}\underbrace{(x-\pi)^{\ell}}_{n}g(t)+\ell Ax^n\underbrace{(x-\pi)^{\ell-1}}_{0}g(t)$$

$t u_x - x u_t = 0$

$x \neq 0$

sii $\quad u_t = \frac{t}{x} u_x$

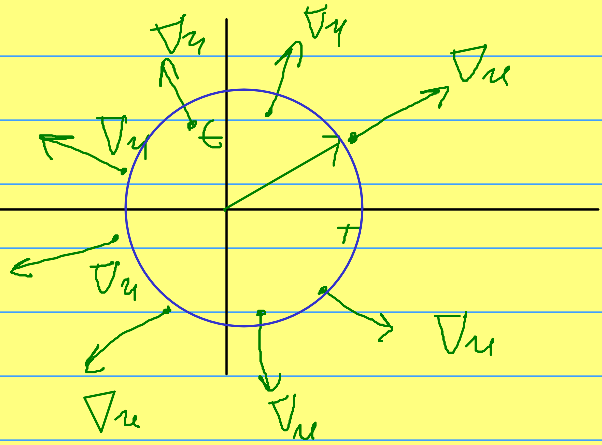
$$\nabla u = (u_x, u_t) = \left(u_x, \frac{t}{x} u_x\right) = u_x \left(1, \frac{t}{x}\right) \parallel u_x (x, t)$$

$u_x \in \mathbb{R}$

$\in \mathbb{R}$

$$\cancel{u_x}(x,t) = x \, \cancel{u_x} \left(1, \frac{t}{x}\right)$$

$\nabla u \parallel (x,t)$



$$\nabla u \parallel (x,t) \Rightarrow u(x,t) = f(x^2 + t^2)$$