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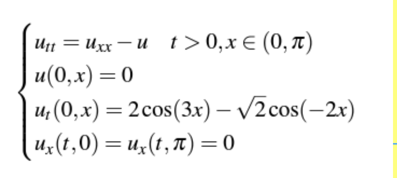


$$\begin{cases} u_{tt} = u_{xx} - u & t > 0, x \in (0,\pi) \\ u(0,x) = 0 \\ u_t(0,x) = 2\cos(3x) - \sqrt{2}\cos(-2x) \\ u_x(t,0) = u_x(t,\pi) = 0 \end{cases}$$

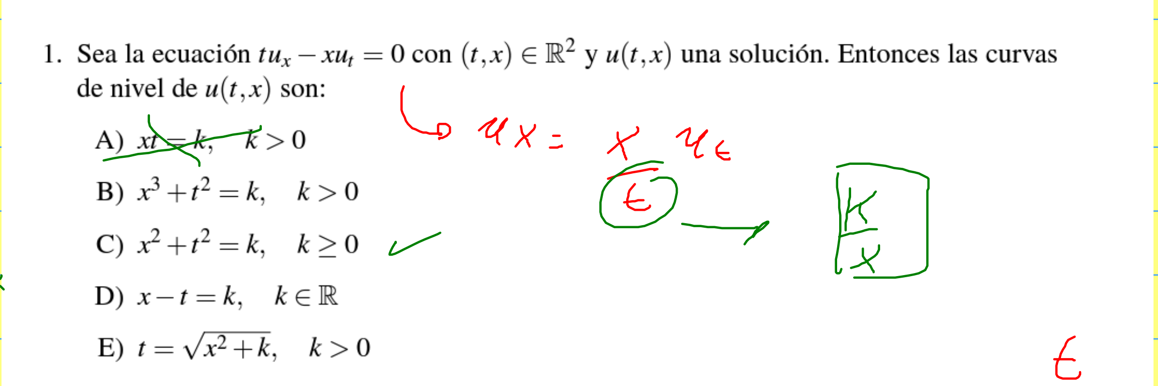
Var. separables

$$\left.\begin{aligned} u(t,x) &= T(t)\,\mathcal{X}(x) \\ u_{tt} &= T''\,\mathcal{X} \\ u_{xx} &= T\,\mathcal{X}'' \end{aligned}\right\} \quad \begin{aligned} T''\mathcal{X} &= T\mathcal{X}'' - T\mathcal{X} \\ &= T(\mathcal{X}'' - \mathcal{X}) \end{aligned}$$

$$\frac{T''}{T}(t) = \frac{\mathcal{X}'' - \mathcal{X}}{\mathcal{X}}(x) = K \rightarrow K \in \mathbb{R}$$

$$\begin{cases} T'' - KT = 0 \\ \mathcal{X}'' - [1+K]\mathcal{X} = 0 \end{cases}$$

$$0 = u(0,x) = T(0)\,\underbrace{\mathcal{X}(x)}_{\rightarrow\ \mathcal{X}(x) = 0\ \forall x\ \rightarrow\ \text{Solución trivial}} \rightarrow T(0) = 0$$



1. Sea la ecuación $tu_x - xu_t = 0$ con $(t,x) \in \mathbb{R}^2$ y $u(t,x)$ una solución. Entonces las curvas de nivel de $u(t,x)$ son:

A) ~~$xt = k$,~~ $k > 0$

B) $x^3 + t^2 = k$, $k > 0$

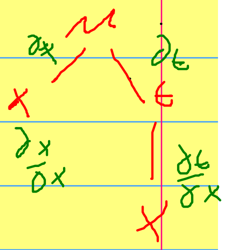
C) $x^2 + t^2 = k$, $k \geq 0$ ✓

D) $x - t = k$, $k \in \mathbb{R}$

E) $t = \sqrt{x^2 + k}$, $k > 0$

$\hookrightarrow u_x = x\, u_t$

$t \rightarrow \boxed{\frac{K}{x}}$



$xt = K$ es curva nivel si $u(x, \overset{t}{\frac{K}{x}}) = ctp$

$$u_x \cdot \frac{\partial x}{\partial x} + u_t \frac{\partial t}{\partial x} = 0$$

$$\frac{x^2}{K} u_t + u_t \left(- \frac{K}{x^2}\right) \neq 0$$

C) $x^2 + t^2 = k, \quad k \geq 0$ → $t = \pm\sqrt{K - x^2}$

$$u_x = \frac{x}{t} u_t$$

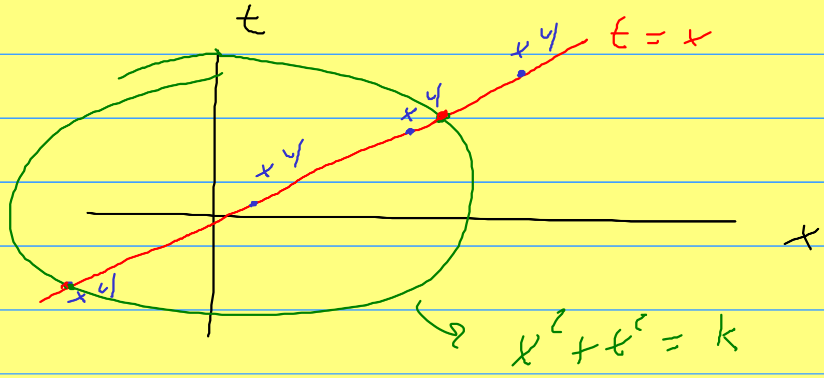
$$u(x, \sqrt{K - x^2}) = cte$$

$$u_x \cdot \overset{1}{\frac{\partial x}{\partial x}} + u_t \frac{\partial(\sqrt{K - x^2})}{\partial x} = 0$$

$$\frac{x}{\sqrt{K - x^2}} u_t + u_t \left(\frac{-x}{\sqrt{K - x^2}}\right) = 0$$

$$\begin{cases} t u_x - x u_t = 0 \\ u(x,x) = x^4 \end{cases}$$

$u(0,0) = 0$



$x^2 + x^2 = K$

$x^2 = \frac{K}{2}$

- $u(x,t) = x^2 t^2$

$$\left.\begin{array}{l} u_x = 2xt^2 \\ u_t = 2x^2 t \end{array}\right\} t u_x - x u_t = 2xt^3 - 2x^3 t \neq 0$$

$$u(x,t) = g(x,t)\,(x^2+t^2)$$

- $x^4 = g(x,x)\,2x^2 \Rightarrow g(x,x) = \frac{x^2}{2}$

$$t u_x - x u_t = 0$$

$$\left.\begin{array}{l} u_x = g_x(2x) = 2x g_x \\ u_t = 2t g_t \end{array}\right\} \begin{cases} 2tx g_x - 2tx g_t = 0 \\ g(x,x) = \frac{x^2}{2} \end{cases}$$

$$2tx\left[g_x - g_t\right] = 0 \qquad \forall (x,t)$$

$$\to \quad g_x - g_t = 0$$

$$g_x = g_t$$

- $X'T = XT' \Rightarrow \frac{X'}{X}(x) = \frac{T'}{T}(t) = \theta \quad \forall (x,t)$

$$\begin{cases} X' - \theta X = 0 \to X' = \theta X \Rightarrow X(x) = A e^{\theta x} \\ T' - \theta T = 0 \to T' = \theta T \Rightarrow T(t) = B e^{\theta t} \end{cases}$$

$$g(x,t) = AB\,e^{\theta(x+t)} \rightsquigarrow g(x,x) = AB e^{2\theta x} \neq \frac{x^2}{2}$$

1. Hallar la solución de la ecuación de ondas,

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \quad (x,t) \in (0,L) \times (0,\infty)$$

con las condiciones de contorno:

$$\begin{cases} u(x,0) = x(L-x) & x \in [0,L] \\ u_t(x,0) = 0 & x \in [0,L] \\ u(0,t) = u(L,t) = 0 & t \in [0,\infty) \end{cases}$$

utilizando el método de separación de variables.

. $u(x,t) = X(x)\,T(t)$, $\dfrac{1}{c^2}\dfrac{T''}{T}(t) = \dfrac{X''}{X}(x)$

$$\begin{cases} T'' - c^2 K T = 0 \\ X'' - K X = 0 \end{cases}$$

$X(0) = X(L) = 0$

$$\begin{cases} X'' - K X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

. Si $K=0$: $X'' = 0 \Rightarrow X(x) = ax + b$

$X(0) = b \to \boxed{b=0}$

$X(L) = aL = 0 \Rightarrow \boxed{a=0}$ $\Big\}$ Trivial

. Si $K>0$: $X'' - KX = 0$

$r^2 - K = 0 \Rightarrow \lambda = \pm\sqrt{K} \in \mathbb{R}$

$$X(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$$

$$0 = X(0) = C_1 + C_2 \longrightarrow C_2 = -C_1$$

$$0 = X(L) = C_1 e^{\sqrt{k}L} + C_2 e^{-\sqrt{k}L}$$

$$\boxed{C_2 = 0}$$

$$0 = C_1 \underbrace{\left[e^{\sqrt{k}L} - e^{-\sqrt{k}L} \right]}_{\overset{\vee}{0}} \Rightarrow \boxed{C_1 = 0}$$

Solución trivial

Lo nuevo:

$K < 0$: $\quad \lambda^2 = K = -|k| \longrightarrow \lambda = \pm \sqrt{|k|}\, i \in \mathbb{C}$

$K = -|k|$ $\qquad\qquad$ $Re(\lambda) = 0$

$$X(x) = C_1 \overset{1}{\boxed{e^{\overset{0}{Re(\lambda)}x}}} \cos(Im(\lambda)x) + C_2 \overset{1}{\boxed{e^{\overset{0}{Re(\lambda)}x}}} \operatorname{sen}(Im(\lambda)x)$$

$$X(x) = C_1 \cos(\sqrt{|k|}\,x) + C_2 \operatorname{sen}(\sqrt{|k|}\,x)$$

$$0 = X(0) = C_1 \longrightarrow \boxed{C_1 = 0}$$

$$0 = X(L) = C_2 \operatorname{sen}(\sqrt{|k|}\,L) \longrightarrow \boxed{C_2 = 0}$$

↳ Solución trivial

$$\operatorname{sen}(\sqrt{|k|}\,L) = 0$$

↳ $\sqrt{|k|}\,L = n\pi$ con $n \in \mathbb{Z}$

$$|K| = \left(\frac{n\pi}{L}\right)^2 \quad , \; K = -\left(\frac{n\pi}{L}\right)^2$$

$$\boxed{X(x) = C_2 \operatorname{sen}\left(\frac{n\pi}{L} x\right)}$$

$$\cdot \; T'' - c^2 K \, T = 0$$

$$\cdot \; T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

$$\lambda^2 + \left(\frac{n\pi c}{L}\right)^2 = 0 \rightarrow \lambda = \pm\left(\frac{n\pi c}{L}\right) i \in \mathbb{C}$$

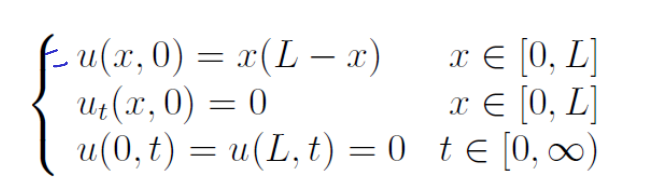
$$\boxed{T(t) = D_1 \cos\left(\frac{n\pi c}{L} t\right) + D_2 \operatorname{sen}\left(\frac{n\pi c}{L} t\right)}$$

$$u(x,t) = X(x)\, T(t)$$
$$= C_2 \operatorname{sen}\left(\frac{n\pi}{L} x\right)\left[D_1 \cos\left(\frac{n\pi c}{L} t\right) + D_2 \operatorname{sen}\left(\frac{n\pi c}{L} t\right)\right]$$

$$\left.\begin{aligned}\tilde{D}_1 &= C_2 D_1 \\ \tilde{D}_2 &= C_2 D_2\end{aligned}\right\} \quad \boxed{u(x,t) = \operatorname{sen}\left(\frac{n\pi}{L} x\right)\left[\tilde{D}_1 \cos\left(\frac{n\pi c}{L} t\right) + \tilde{D}_2 \operatorname{sen}\left(\frac{n\pi c}{L} t\right)\right]}$$

u_0 –

v_0 =



$$\begin{cases} u(x,0) = x(L-x) & x \in [0,L] \\ u_t(x,0) = 0 & x \in [0,L] \\ u(0,t) = u(L,t) = 0 & t \in [0,\infty) \end{cases}$$

$$u(x,0) = \tilde{D}_1 \operatorname{sen}\left(\frac{n\pi}{L} x\right) \overset{\text{queremos}}{=} x(L-x)$$

Teorema 0.3.

Sea $\overset{x(L-x)}{u_0} = \sum_{k=1}^{+\infty} b_k sen\left(\frac{k\pi}{L}x\right)$ *y* $v_0 = \sum_{k=1}^{+\infty} b'_k sen\left(\frac{k\pi}{L}x\right)$ *las condiciones iniciales del problema (0.10). Si*

$$|b_k| < \frac{M}{k^4} \qquad |b'_k| < \frac{N}{k^3} \qquad N, M \in \mathbb{R}$$

entonces:

$$\rightarrow \quad U(t,x) = \sum_{k=1}^{+\infty} sen\left(\frac{k\pi}{L}x\right)\left(A_k cos\left(\frac{k\pi c}{L}t\right) + B_k sen\left(\frac{k\pi c}{L}t\right)\right)$$

con $A_k = b_K$ *y* $B_k = b'_k \frac{L}{k\pi c}$ *es solución al problema (0.10).*

En nuestro caso $u_0 = u(x,0) = x(L-x)$

$v_0 = u_t(x,0) = 0 \Rightarrow b'_K = 0 \quad \forall K$

Queremos calcular la serie de $\underbrace{\text{Fourier de senos}}$ de $x(L-x)$:

→ Extensión impar

$$b_K = \frac{2}{L}\int_0^L x(L-x)\, sen\left(\frac{K}{L}\pi x\right)dx$$

$$= \frac{2}{L}\left[\int_0^L L\overset{f}{x}\,\overbrace{sen\left(\frac{K\pi}{L}x\right)}^{g'}dx - \int_0^L x^2 sen\left(\frac{K\pi}{L}x\right)dx\right]$$

$$= \frac{2}{L}\left[L\left(-\frac{x\cos\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}\Bigg|_0^L\right) + L\int_0^L \frac{\cos\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}dx \right.$$
$$\left. - \int_0^L x^2 sen\left(\frac{K\pi}{L}x\right)dx \qquad 0 \right]$$

$$= \frac{2}{L}\left[\frac{L^2(-1)^{K+1}}{K\pi} + \boxed{\frac{L^2}{K\pi}\frac{sen\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}\Bigg|_0^L} \right.$$
$$\left. - \int_0^L x^2 sen\left(\frac{K\pi}{L}x\right)dx\right]$$

$$= \frac{2}{L}\left[\frac{L^3}{K\pi}(-1)^{K+1} \quad - \; - \right.$$

$$\frac{L^3}{K\pi}(-1)^{K+1}$$

$$\int_0^L \overset{f}{x^2}\, \operatorname{sen}\left(\overset{g'}{\frac{K\pi}{L}x}\right) dx = \boxed{-\frac{x^2 \cos\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}\Bigg|_0^L} + \int_0^L \frac{2x\cos\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}$$

$$= \frac{L^3}{K\pi}(-1)^{K+1} + \frac{2L}{K\pi}\int_0^L \overset{f}{x}\, \cos\left(\overset{g'}{\frac{K\pi}{L}x}\right) dx$$

$$= \frac{L^3}{K\pi}(-1)^{K+1} + \frac{2L}{K\pi}\left[\cancelto{0}{\frac{x\operatorname{sen}\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}}\Bigg|_0^L - \int_0^L \frac{\operatorname{sen}\left(\frac{K\pi}{L}x\right)dx}{\frac{K\pi}{L}} \right]$$

$$= \frac{L^3}{K\pi}(-1)^{K+1} + 2\left(\frac{L}{K\pi}\right)^2 \frac{\cos\left(\frac{K\pi}{L}x\right)}{\frac{K\pi}{L}}\Bigg|_0^L$$

$$= \frac{L^3}{K\pi}(-1)^{K+1} + 2\left(\frac{L}{K\pi}\right)^3\left[(-1)^K - 1\right]$$

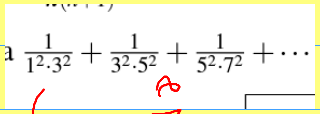
$$b_k = \frac{2}{\cancel{L}}\left[\frac{L^{\cancel{2}}}{k\pi}(-1)^{k+1} - \frac{L^{\cancel{3}\,2}}{k\pi}(-1)^{k+1} - 2\left\{\frac{L^2}{(k\pi)}\right\}^3\left((-1)^k-1\right)\right]$$

$$b_k = \frac{2}{k\pi}\left(L(-1)^{k+1} - L^2(-1)^{k+1} - \frac{2L^2}{(k\pi)^3}\left((-1)^k-1\right)\right)$$

$$|b_k| \leq \frac{2}{k\pi}\left[\overset{\nearrow 0}{L} + \overset{\nearrow 0}{L^2} + \overset{\nearrow 0}{\frac{2L^2}{(k\pi)^3}}\,|-1|\right]$$

$$\leq \frac{2}{k\pi}\frac{4L^2}{(k\pi)^3} = \frac{1}{k^4}\overset{M}{\boxed{\frac{8L^2}{\pi^4}}}$$

$$\boxed{|b_k| \leq \frac{M}{k^4}}$$



a $\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\cdots$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2(n+2)^2}$$

$$\frac{1}{(2n-1)^2\,(2n+1)^2}$$

d) $a_n(x) = (-1)^n \dfrac{x^{2n+1}}{2n+1!}$

$x \in \mathbb{R}$

- $S_n = \sum_{j=1}^{n} a_j(x)$.

$|a_j(x)| < A_j$, $\sum A_j$ converge

$\sum_{j=1}^{+\infty} a_j(x)$ converge uniformemente

- $\underline{x_0 \text{ fijo}}$: $\sum_{n=1}^{+\infty} (-1)^n \dfrac{x_0^{2n+1}}{(2n+1)!}$.

Si $|x_0| < 1$

$\underline{x_0 = 1}$: $-\dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \dfrac{1}{9!} - \dots$

$\sum_{n=1}^{+\infty} \dfrac{(-1)^n}{(2n+1)!}$ converge

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{(-1)^n}(-1)}{(2(n+1)+1)!} \frac{(2n+1)!}{\cancel{(-1)^n}}$$

$$= -\frac{(2n+1)!}{(2n+2+1)!} = -\frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}$$

$$= \left|-\frac{1}{(2n+3)(2n+2)}\right| = \frac{1}{(2n+3)(2n+2)} < 1$$

$$a_n = (-1)^n \frac{x_0^{2n+1}}{(2n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\overset{-1}{\cancel{(-1)^{n+1}}} \; x_0^{\overset{2}{\cancel{2n+3}}}}{(\cancel{2n+3})! \atop (2n+3)(2n+2)} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-1)^n} \; \cancel{x_0^{2n+1}}}$$

$$= - \frac{x_0^2}{(2n+3)(2n+2)} < 1$$

↳ A partir de cierto n_0

$$\cdot \; \lim_{n} \; \sup_{x \in \mathbb{R}} \Big\{ \sum_{j=1}^{n} a_j(x) - \sum_{j=1}^{+\infty} a_j(x) \Big\}$$