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10. *a*) Hallar la serie de Fourier de la función 2π-peródica definida como:

$$f(x) = \begin{cases} 0 & \text{si} \quad -\pi \leqslant x < 0 \\ x & \text{si} \quad 0 \leqslant x < \pi \end{cases}.$$

$$S_\infty(x) = \frac{a_0}{2} + \sum_{K=1}^{\infty} a_K \cos Kx + b_K \operatorname{sen} Kx$$

$$\left.\begin{aligned} a_0 &= \langle f, 1 \rangle \\ a_K &= \langle f, \cos Kx \rangle \\ b_K &= \langle f, \operatorname{sen} Kx \rangle \end{aligned}\right\} \quad \langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

En general: para funciones $2L$ periódicas

$$\langle f, g \rangle = \frac{1}{L}\int_{-L}^{L} f(x)\, g(x)\, dx$$

$$a_K = \langle f, \cos\left(\tfrac{K\pi}{L}x\right) \rangle$$
$$b_K = \langle f, \operatorname{sen}\left(\tfrac{K\pi}{L}x\right) \rangle$$

10. *a*) Hallar la serie de Fourier de la función 2π-peródica definida como:

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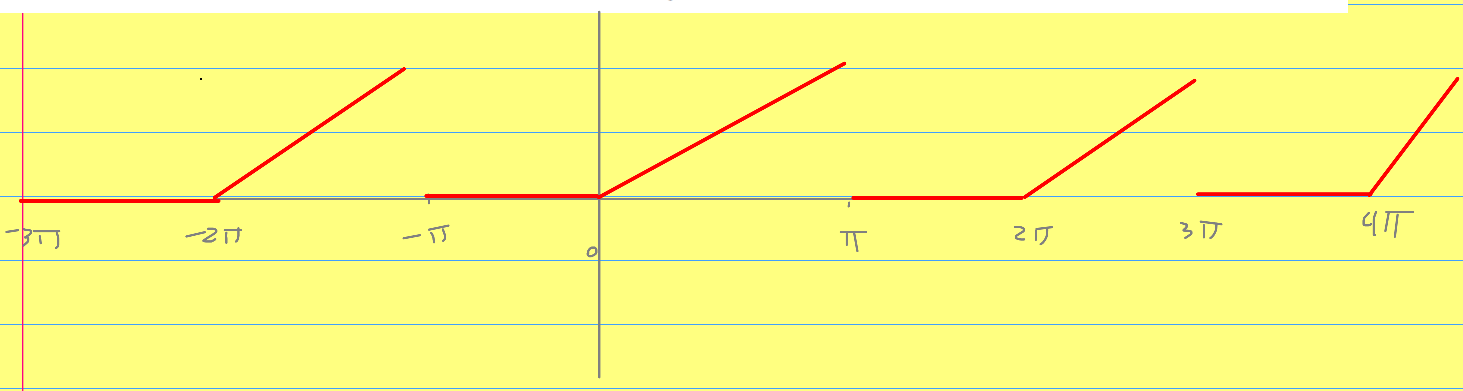


$$a_0 = \langle f, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\overbrace{\cancel{\int_{-\pi}^{0} 0\,dx}}^{0} + \int_{0}^{\pi} x\,dx\right]$$

$$f(x) = \begin{cases} 0 & si \quad -\pi \leq x < 0 \\ x & si \quad 0 \leq x < \pi \end{cases}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{1}{\pi}\,\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_K = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos Kx\,dx = \frac{1}{\pi}\left[\overbrace{\cancel{\int_{-\pi}^{0} 0\cdot\cos Kx\,dx}}^{0} + \int_0^{\pi} x\cos Kx\,dx\right]$$

Partes

$$a_K = \frac{1}{\pi}\int_0^{\pi} \overset{g}{x}\,\overbrace{\cos Kx}^{f'}\,dx = \frac{1}{\pi}\left[\overbrace{\cancel{\frac{x\,\text{sen}\,Kx}{K}\Big|_0^{\pi}}}^{0} - \int_0^{\pi}\frac{\text{sen}\,Kx}{K}\,dx\right]$$

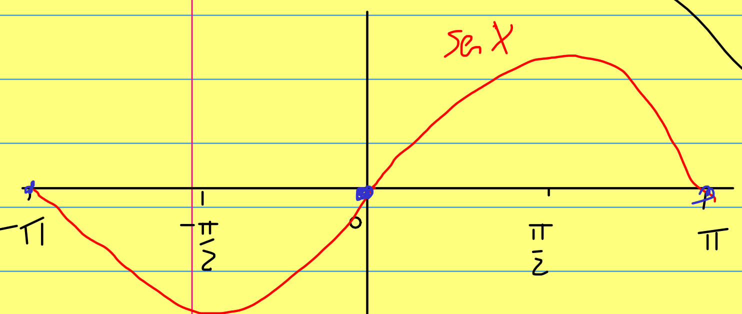
$$\boxed{\begin{aligned} g &= x \to g' = 1 \\ f' &= \cos Kx \to f = \frac{\text{sen}\,Kx}{K} \end{aligned}}$$

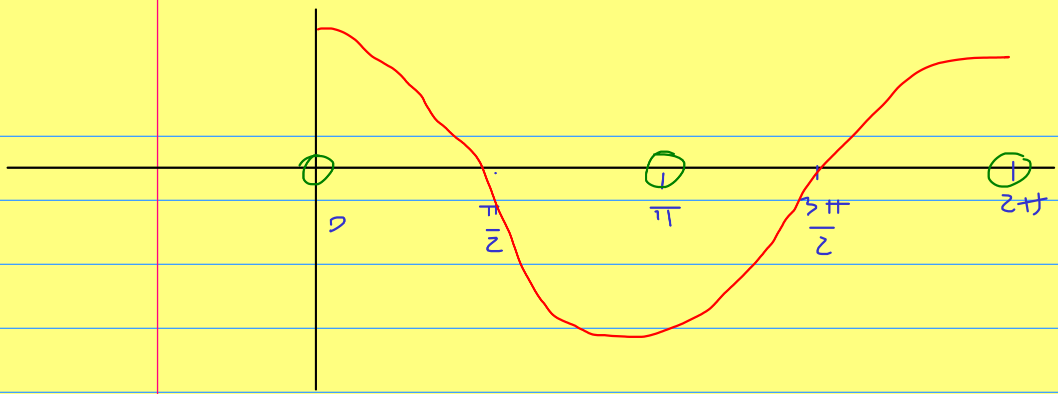
$$= -\frac{1}{K\pi}\int_0^{\pi} \text{sen}\,Kx\,dx$$

$$= \frac{1}{K^2\pi}\left(\cos Kx\Big|_0^{\pi}\right)$$

$$= \frac{1}{K^2\pi}\left(\overbrace{\cos K\pi}^{(-1)^K} - \overbrace{\cos 0}^{1}\right)$$

$$= \frac{(-1)^K - 1}{K^2\pi}$$





$$\cos k\pi = \begin{cases} 1 & \text{si } k \text{ es par} \\ -1 & \text{si } k \text{ es impar} \end{cases} = (-1)^k$$

$$\boxed{a_k = \frac{(-1)^k - 1}{k^2 \pi}}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \overset{g}{x}\, \overbrace{\operatorname{sen} kx}^{s'}\, dx = \frac{1}{\pi}\left[-\frac{x\cos kx}{k}\Bigg|_0^{\pi} - \int_0^{\pi} -\frac{\cos kx}{k}\, dx \right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{k}\overbrace{\cos k\pi}^{(-1)^k} - \underbrace{\cancel{\frac{0\cos 0}{k}}}_{0} + \frac{1}{k}\underbrace{\cancel{\int_0^{\pi} \cos kx\, dx}}_{0} \right]$$

$$\int_0^{\pi} \cos kx\, dx = \frac{1}{k}\operatorname{sen} kx \Big|_0^{\pi} = 0$$

$$b_k = -\frac{(-1)^k}{k} = \frac{(-1)^{k+1}}{k} \rightarrow \boxed{b_k = \frac{(-1)^{k+1}}{k}}$$

Entonces la serie de Fourier de

$$f(x) = \begin{cases} 0 & \text{si} \quad -\pi < x < 0 \\ x & \text{si} \quad 0 < x < \pi \end{cases}$$ es:

$$\boxed{S_\infty(x) = \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{(-1)^K - 1}{K^2\pi} \cos Kx + \frac{(-1)^{K+1}}{K} \operatorname{sen} Kx}$$

$$a_K = \frac{2}{\pi}\int_0^{\pi} f(x)\cos Kx\, dx$$

f es par ⇒ f sen Kx es impar ⇒ $b_K = 0$

f es impar ⇒ f cos Kx es par ⇒ $a_K = 0$

$$b_K = \frac{2}{\pi}\int_0^{\pi} f(x)\operatorname{sen} Kx\, dx$$

• Si g es par ⇒ $\int_{-a}^{a} g(x)\,dx = 2\int_0^a g(x)\,dx$

b) Sustituyendo x por π, demostrar que:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$S(\pi) = \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{(-1)^K - 1}{K^2\pi} \boxed{\overset{(-1)^K}{\cos K\pi}} + \boxed{\overset{0}{\overset{||}{\frac{(-1)^{K+1}}{K} \operatorname{sen} K\pi}}}$$

$$S(\pi) = \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{\boxed{(-1)^K(-1)^K} - (-1)^K}{K^2\pi}$$

$$(-1)^{2K} = 1$$

$$S(\pi) = \frac{\pi^2}{4} + \sum_{K=1}^{\infty} \frac{1-(-1)^K}{K^2\pi}$$

b) Sustituyendo x por π, demostrar que:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

• Recordar: (Teo. Riemann para series)

Si $\sum_{K=1}^{\infty} a_K$ converge absolutamente, entonces no importa en que orden sumemos.

• $a_K = \frac{1-(-1)^K}{K^2\pi} \geq 0$, entonces

$\sum_{K=1}^{\infty} \frac{1-(-1)^K}{K^2\pi}$ es abs. convergente:

$$\Rightarrow \sum_{K=1}^{\infty} \frac{1-(-1)^K}{K^2\pi} = \underbrace{\sum_{K\,par} \frac{1-(-1)^K}{K^2\pi}}_{=0} + \sum_{K\,impar} \frac{1-(-1)^K}{K^2\pi}$$

$$S(\pi) = \frac{\pi^2}{4} + \frac{2}{\pi} \sum_{K\,impar} \frac{1}{K^2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}} = \frac{\pi^2}{8}.$$

• Si K es impar $\Leftrightarrow$ $K = 2n-1$ para algún n

Entonces: $\displaystyle\sum_{K \text{ impar}} \frac{1}{K^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

$$S(\pi) = \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{S(\pi) - \frac{\pi}{4}}{\frac{2}{\pi}}$$

$$S(\pi) = \frac{\pi}{2} \rightarrow S(\pi) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

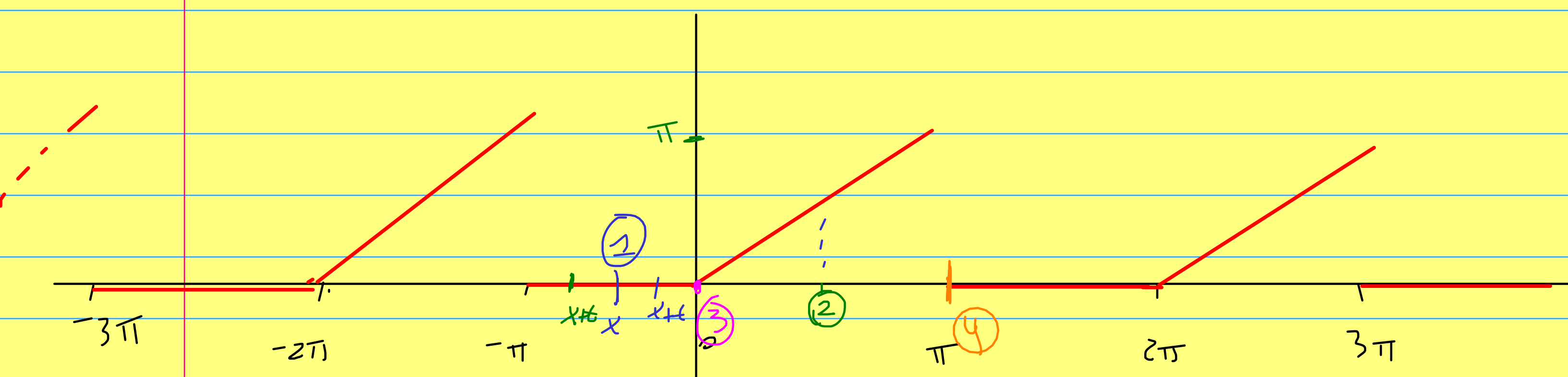
$$\frac{S(\pi) - \frac{\pi}{4}}{\frac{2}{\pi}} = \frac{\frac{\pi}{4}}{\frac{2}{\pi}} = \frac{\pi}{4} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}$$

Dini: $f:\mathbb{R}\to\mathbb{R}$ continua a trozos tq $\forall x\in\mathbb{R}$ existen (y son finitos) los siguientes límites:

$$\lim_{t\to 0^+} \frac{f(x+t)-f(x^+)}{t} \quad \left\{ \quad \lim_{t\to 0^-} \frac{f(x+t)-f(x^-)}{t}\right.$$

Entonces $\forall x\in\mathbb{R}$.

$$f(x) = \lim_n S_n(x) \xrightarrow{\text{C.P.}} \frac{f(x^-)+f(x^+)}{2}$$



1 $f(x+t) = 0 = f(x) \Rightarrow \lim_{t\to 0^\pm} \frac{f(x+t)-f(x^\pm)}{t} = \lim_{t\to 0^\pm}\frac{0}{t} = 0$

2 ✓

3 ✓

4 ✓

Entonces Dini nos dice

$$f(\pi) = \frac{\overset{\pi}{\overline{f(\pi^-)}} + \overset{0}{\overline{f(\pi^+)}}}{2} = \frac{\pi}{2} \Rightarrow \boxed{f(\pi)=\frac{\pi}{2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

c) A partir de lo anterior, concluir que:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$a_n = \frac{1}{n^2} > 0 \rightarrow |a_n| = a_n$

Como $\sum_{n=1}^{\infty} \frac{1}{n^2}$ es abs. convergente:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ par}} \frac{1}{n^2} + \sum_{n \text{ impar}} \frac{1}{n^2}$$

$$= \sum_{K=1}^{\infty} \frac{1}{(2K)^2} + \boxed{\sum_{K=1}^{\infty} \frac{1}{(2K-1)^2}}$$

$$\frac{\pi^2}{8}$$

$$= \frac{1}{4} \sum_{K=1}^{\infty} \frac{1}{K^2}$$

$$\sum_{K=1}^{\infty} \frac{1}{K^2} = \frac{1}{4} \sum_{K=1}^{\infty} \frac{1}{K^2} + \frac{\pi^2}{8}$$

$$\frac{3}{\cancel{4}} \sum_{K=1}^{\infty} \frac{1}{K^2} = \frac{\pi^2}{\cancel{8}\,2} \rightarrow \boxed{\sum_{K=1}^{\infty} \frac{1}{K^2} = \frac{\pi^2}{6}}$$