27/10

5. Sea la sucesión $f_n(x) = \dfrac{2^n x}{1 + n2^n x^2}$ para $x \in [0, 1]$.

c) Calcular el supremo de $f_n(x)$ para $x \in [0, 1]$.

$$f_n'(x) = \frac{2^n(1 + n2^n x^2) - 2^n x(2n2^n x)}{(1 + n2^n x^2)^2}$$

$$= \frac{2^n + \cancel{n2^{2n} x^2} - \cancel{2}n2^{2n} x^2}{(1 + n2^n x^2)^2}$$

$$f_n'(x) = \frac{2^n - n2^{2n}x^2}{(1 + n2^n x^2)^2} = \frac{2^n[1 - n2^n x^2]}{(1 + n2^n x^2)^2}$$

$x \in [0, 1]$

$$f_n'(x) = 0 \iff 1 - n2^n x^2 = 0$$

$$x^2 = \frac{1}{n2^n} \to \boxed{x = \pm \frac{1}{\sqrt{n2^n}}}$$

Los candidatos a supremo son: $\dfrac{1}{\sqrt{n2^n}}$, 0, 1

$f_n(x) = \dfrac{2^n x}{1 + n2^n x^2}$ -

$$f_n(0) = 0$$

$$f_n(1) = \frac{2^n}{1 + n2^n}$$

$$f_n\left(\frac{1}{\sqrt{n2^n}}\right) = \frac{2^n \frac{1}{\sqrt{n2^n}}}{1 + n2^n \cdot \frac{1}{n2^n}} \overset{=1}{} = \frac{2^n}{2\sqrt{n2^n}}$$

$$f_n(1) = \frac{2^n}{1+n2^n}$$

$$\boxed{f_n\left(\frac{1}{\sqrt{n2^n}}\right) = \frac{2^n}{2\sqrt{n2^n}}}$$

Como ambos numeradores son iguales, alcanza con estudiar los denominadores, el número más grande será el que tiene el denominador más chico.

$$(1+n2^n)^2 \qquad \left(2\sqrt{n2^n}\right)^2$$

$$= \qquad\qquad = 4n2^n$$

$$1 + 2n2^n + n^2 2^{2n}$$

$$1 + 2n2^n + n^2 2^{2n} \quad - \quad \overset{2^2}{4}\, n 2^n = 2^2 n 2^n$$

$$1 + \cancel{2n}2^n + n^2 2^{2n} - 2^2 \cancel{n2}^n = 1 - 2 + n^2 2^{2n}$$

$$= -1 + n^2 2^{2n} > 0$$

$$\forall n \in \mathbb{N}$$

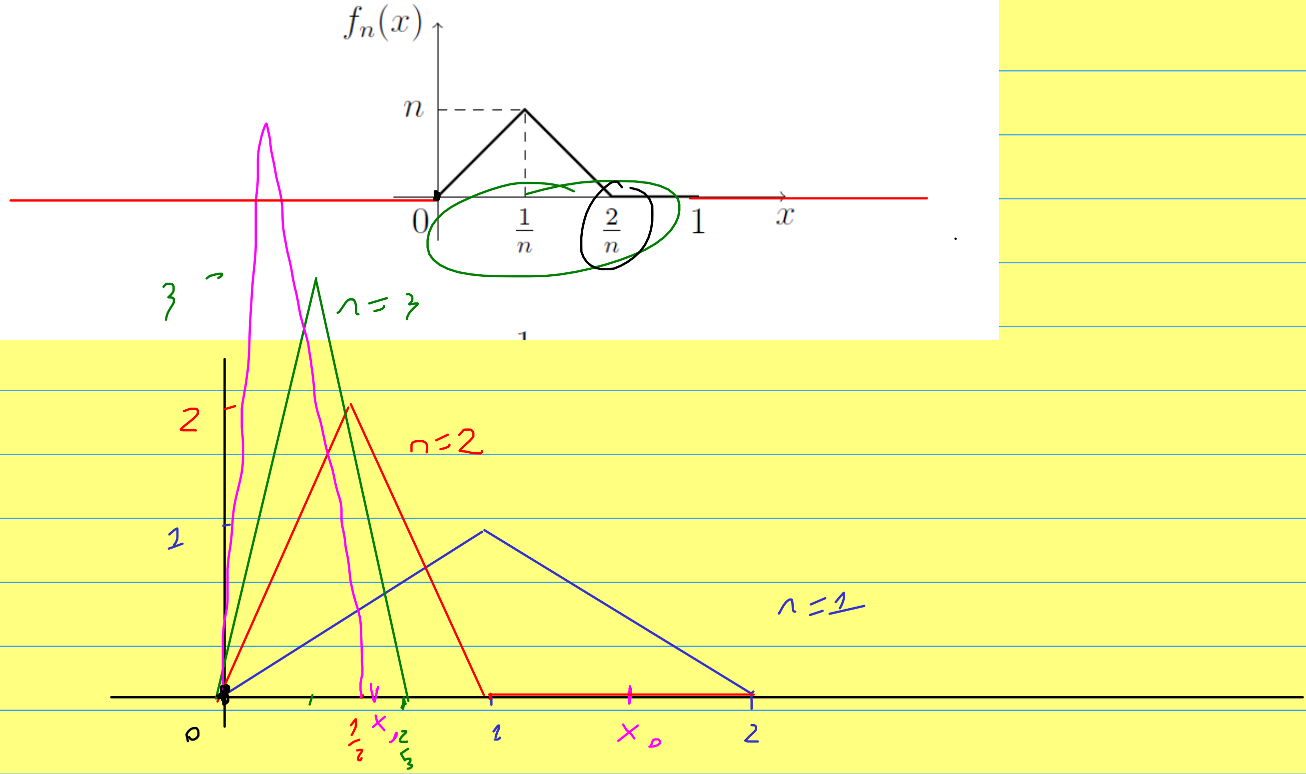
$f(1)$ tiene denominador más grande, entonces $f\left(\frac{1}{\sqrt{n2^n}}\right)$ es el supremo.

$$\boxed{\sup_{x\in[0,1]} f_n = \frac{2^n}{2\sqrt{n2^n}}}$$

$$\frac{\sqrt{2^{2n}}}{2\sqrt{n}\sqrt{2^n}} = \frac{1}{2\sqrt{n}}\sqrt{\frac{2^{2n}}{\cancel{2^n}}}$$

$$\frac{1}{2\sqrt{n}} \cdot \sqrt{2^n} = \sqrt{\frac{2^n}{4n}}$$

6. Sea $f_n : [0,1] \to \mathbb{R}$ definida como en la figura:



• Quiero ver que $f_n \xrightarrow{c.p.} 0$

$x_0 \in \mathbb{R}$, quiero ver si $\lim\limits_n |f_n(x_0) - 0| = 0$

$\Updownarrow$

$\lim\limits_n f_n(x_0) = 0$

Si $x_0 \notin [0,2]$, $f_n(x_0) = 0$ $\forall n$, por lo tanto $f_n(x_0) \to 0$

• Si $x_0 \in (0,2]$: Obs. $f_n(x_0) = 0$ $\forall n \geq n_0$ si x_0 no está en las bases de los triángulos a partir de n_0

. $x_0 > 0$

La base del triángulo definido por f_n es $\left[0, \frac{2}{n}\right]$

x_0 no está en la base si $x_0 \notin \left[0, \frac{2}{n}\right]$

$x_0 \notin \left[0, \frac{2}{n}\right]$ sii $x_0 > \frac{2}{n}$

. Como $\frac{2}{n} \xrightarrow[n]{} 0 \Rightarrow$ existe $n_0 \in \mathbb{N}$ tq $\frac{2}{n} < x_0 \quad \forall n \geq n_0$

. Es decir $\forall n \geq n_0$, $x_0 \notin \left[0, \frac{2}{n}\right]$

. Es decir $\forall n \geq n_0$, x_0 no está en la base de los triángulos

. Es decir $\boxed{\forall n \geq n_0 \quad f_n(x_0) = 0}$

$\lim_n f_n(x_0) = 0$

. Si $x_0 = 0$, $f_n(x_0) = 0 \quad \forall n \in \mathbb{N} \Rightarrow \lim_n f_n(0) = 0$

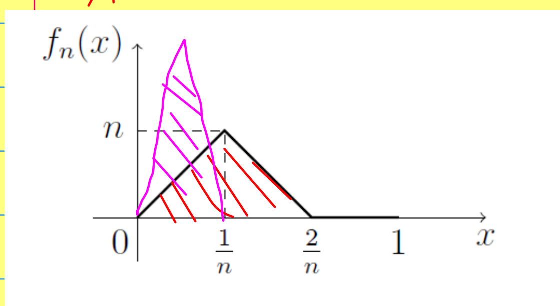
. $f_n \xrightarrow{c.p.} 0$

b) Probar que $\overset{1}{\overset{\shortparallel}{\lim \int_0^1 f_n}} \neq \overset{0}{\overset{=}{\underbrace{\int_0^1 \lim f_n}}}$.

$$\int_0^1 \underbrace{\lim_n f_n(x)}_{\overset{\shortparallel}{0}}\, dx = \int_0^1 0\, dx = 0$$

base altura

$$\int_0^1 f_n(x)\, dx. = \frac{\frac{2}{n} \cdot n}{2} = \frac{2n}{2n} = 1 \qquad \forall n$$



$$\int_0^1 f_n(x)\, dx = 1 \qquad \forall n$$

$$\lim_n \int_0^1 f_n(x)\, dx = 1$$

c) Usar que si $f_n \rightrightarrows f$ entonces

$$\lim_n \int_a^b f_n(x)\, dx = \int_a^b \lim_n f_n(x)\, dx$$

Fourier

- $V = \{ f : \mathbb{R} \to \mathbb{R}$ continua a trozos, 2π periódicas$\}$

Continua salvo en finitos puntos y donde es discontinua los límites laterales existen.

- $f, g \in V \Rightarrow \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$

- $S = \{ \frac{1}{\sqrt{2}}, \cos(x), \operatorname{sen}(x), \cos(2x), \operatorname{sen}(2x), \dots \}$

Conjunto ortonormal con $\langle f, g \rangle$

- Definimos la suma parcial de Fourier de la función $f \in V$ como:

$$S_n(x) = \frac{a_0}{2} + \sum_{K=1}^{n} \underbrace{\langle f, \cos Kx \rangle}_{a_K} \cos Kx + \sum_{K=1}^{n} \underbrace{\langle f, \operatorname{sen} Kx \rangle}_{b_K} \operatorname{sen} Kx$$

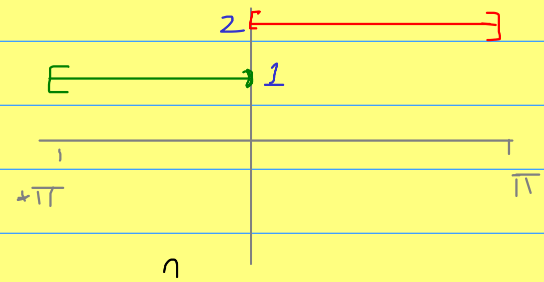
$a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle$

. En caso de que las sumas finitas converja, definimos la serie de Fourier como:

$$S_\infty(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \langle f, \cos kx \rangle \cos kx + \sum_{k=1}^{\infty} \langle f, \operatorname{sen} kx \rangle \operatorname{sen} kx$$

En los siguientes casos hallar la serie ...

a) $f(x) = \begin{cases} 1 & \text{si } -\pi < x < 0 \\ 2 & \text{si } 0 \leq x \leq \pi \end{cases}$



$$\langle f, \tfrac{1}{\sqrt{2}} \rangle$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \langle f, \cos kx \rangle \cos kx + \sum_{k=1}^{n} \langle f, \operatorname{sen} kx \rangle \operatorname{sen} kx$$

. $a_0 = \langle f, \frac{1}{\sqrt{2}} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}} \, dx$

$$= \frac{1}{\sqrt{2}\pi} \left[\int_{-\pi}^{0} \overset{x}{1} \, dx + \int_{0}^{\pi} \overset{2x}{2} \, dx \right]$$

$$= \frac{1}{\sqrt{2}\pi} \left[x \Big|_{-\pi}^{0} + 2x \Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\sqrt{2}\pi} \left[\pi + 2\pi \right] = \frac{3\pi}{\sqrt{2}\pi} = \frac{3}{\sqrt{2}}$$

$$f(x) = \begin{cases} 1 & \text{si } -\pi \le x < 0 \\ 2 & \text{si } 0 \le x \le \pi \end{cases}$$

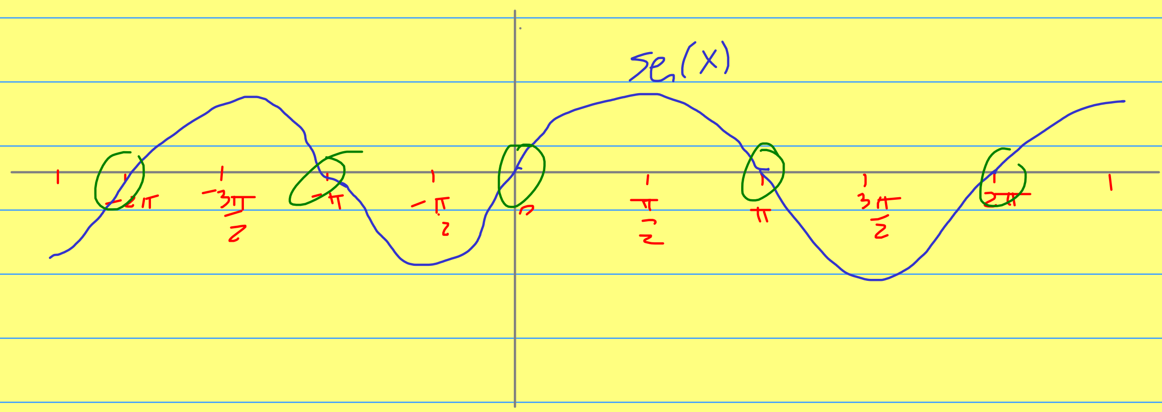
$$a_K = \langle f, \cos Kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos Kx \; dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \cos Kx \, dx + \int_{0}^{\pi} 2 \cos Kx \; dx \right]$$

$$= \frac{1}{\pi} \left[\left. \frac{\operatorname{sen} Kx}{K} \right|_{-\pi}^{0} + 2 \left. \frac{\operatorname{sen} Kx}{K} \right|_{0}^{\pi} \right]$$

0 $\quad$ 0

$$\boxed{\text{Primitiva de } \cos Kx : \frac{\operatorname{sen}(Kx)}{K}}$$



Entonces $a_K = 0 \quad \forall K \geq 1$

$$b_K = \langle f, \operatorname{sen} Kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen} Kx \; dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \operatorname{sen} Kx \; dx + \int_{0}^{\pi} 2 \operatorname{sen} Kx \; dx \right]$$

$$= \frac{1}{\pi} \left[- \left. \frac{\cos(Kx)}{K} \right|_{-\pi}^{0} - \left. \frac{2 \cos(Kx)}{K} \right|_{0}^{\pi} \right]$$

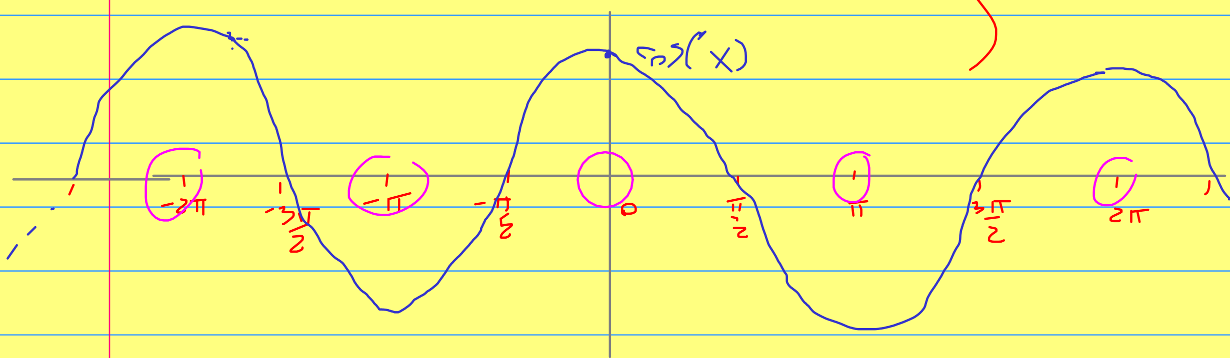
$$\boxed{\text{Primitiva de } \operatorname{sen} Kx : -\frac{\cos(Kx)}{K}}$$

$=$

$$= \frac{1}{\pi}\left[\cancel{-\frac{1}{k}} + \frac{\overbrace{\cos(-k\pi)}^{\cos(k\pi)}}{k} - \frac{\cancel{2}\cos(k\pi)}{k} + \frac{1\cancel{2}}{k} \right]$$

$$= \frac{1}{\pi}\left[\frac{1}{k} - \frac{\cos(k\pi)}{k} \right]$$

$$\cos(k\pi) = \begin{cases} 1 & \text{si } k \text{ es par} \\ -1 & \text{si } k \text{ es impar} \end{cases}$$



$$b_k = \frac{1}{\pi}\left[\frac{1}{k} - \frac{\cos k\pi}{k} \right]$$

$$\cos(k\pi) = \begin{cases} 1 & \text{si } k \text{ es par} \\ -1 & \text{si } k \text{ es impar} \end{cases}$$

K par: $b_k = \frac{1}{\pi}\left[\cancel{\frac{1}{k}} - \cancel{\frac{1}{k}} \right] = 0$

K impar $b_k = \frac{1}{\pi}\left[\frac{1}{k} - \frac{-1}{k} \right] = \frac{2}{\pi k}$

$$S_n(x) = \frac{3}{2\sqrt{2}} + \sum_{k=1}^{n} b_k \operatorname{sen} kx = \frac{3}{2\sqrt{2}} + \sum_{\substack{k=1 \\ k \text{ impar}}}^{n} \frac{2}{\pi k} \operatorname{sen} kx$$

Entonces la serie parcial de Fourier de f es:

$$\boxed{S_n(x) = \frac{3}{2\sqrt{2}} + \sum_{\substack{K=1 \\ K \text{ impar}}}^{n} \frac{2}{\pi K} \operatorname{sen} Kx}$$

- La Serie de Fourier es:

$$\lim_{n} S_n(x) = \frac{3}{2\sqrt{2}} + \sum_{\substack{K=1 \\ K \text{ impar}}}^{\infty} \frac{2}{\pi K} \operatorname{sen} Kx$$