8/9

1. $A_1 = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

2. $A_2 = \begin{pmatrix} 0 & -1 \\ 8 & -6 \end{pmatrix}$

3. $A_3 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$

4. $A_4 = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$

A_1: $(1-\lambda)(-1-\lambda) - 3 = 0$

$\lambda^2 - 1 - 3 = 0$

$\lambda^2 - 4 = 0 \to \lambda = \pm 2$

A_2: $-\lambda(-6-\lambda) + 8 = 0$

$\lambda^2 + 6\lambda + 8 = 0$

$\sqrt{36 - 32} = \sqrt{4} = 2 > 0$

A_3 $-\lambda(-1-\lambda) + 2 = 0$

$\lambda^2 + \lambda + 2 = 0$

$\sqrt{1 - 8}$ complejo

A_4: $(1-\lambda)(-1-\lambda) + 10 = 0$

$\lambda^2 + 10 = 0$

$\lambda = \pm\sqrt{-10}$ complejo

2. $A_2 = \begin{pmatrix} 0 & -1 \\ 8 & -6 \end{pmatrix}$

$p(\lambda) = -\lambda(-6-\lambda) + 8$

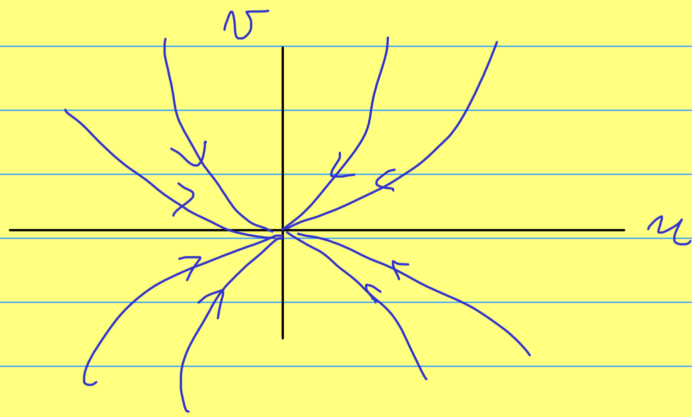
$\lambda^2 + 6\lambda + 8$

$\lambda = \dfrac{-6 \pm \sqrt{36 - 32}}{2} = -3 \pm 2 \to -2, \; -4$

$D = \begin{pmatrix} \overset{\mu}{-2} & 0 \\ 0 & \underset{\lambda}{-4} \end{pmatrix}$

$\dot{y} = Dy$

$\dfrac{\lambda}{\mu} = 2$



$\underline{P}$:

2. $A_2 = \begin{pmatrix} 0 & -1 \\ 8 & -6 \end{pmatrix}$

$\lambda = -4$

$\lambda = -2$

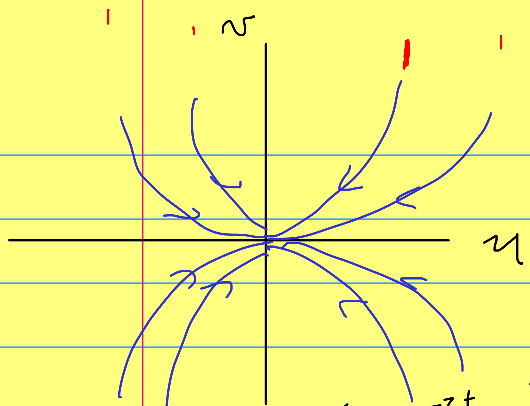
$\underline{v_1}$: $A_2 + 2I = \begin{pmatrix} 2 & -1 \\ 8 & -4 \end{pmatrix}$

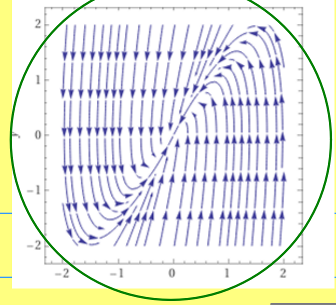
$2x - y = 0 \to y = 2x \to \boxed{v_1 = (1,2)}$

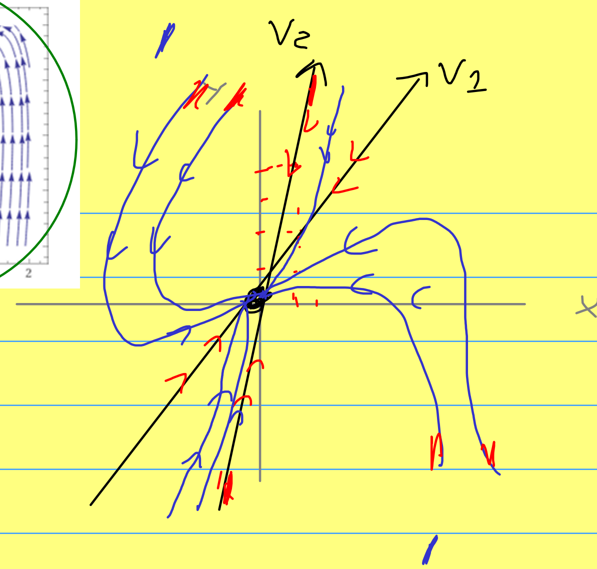
$\underline{v_2}$: $A_2 + 4I = \begin{pmatrix} 4 & -1 \\ 8 & -2 \end{pmatrix} \to 4x - y = 0 \to y = 4x$

$\boxed{v_2 = (1,4)}$

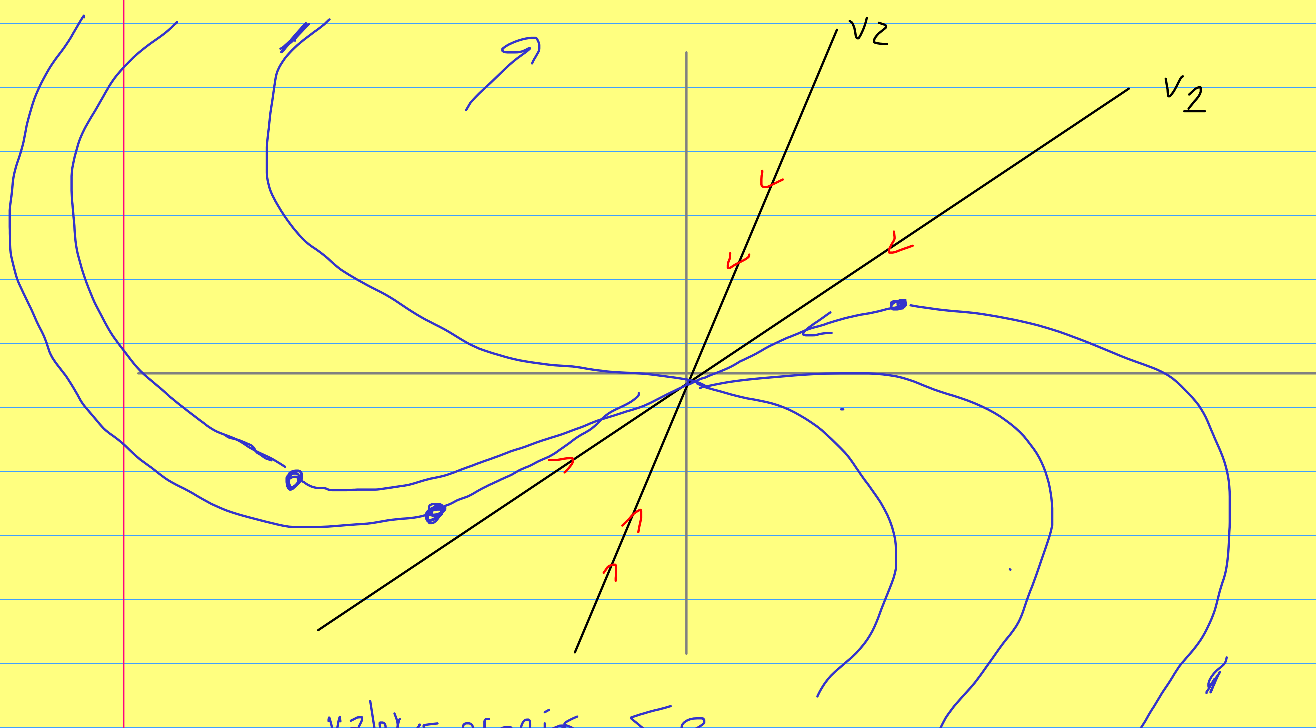
$P = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$







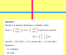
$$y(t) = \begin{pmatrix} u_0 e^{-2t} \\ v_0 e^{-4t} \end{pmatrix}$$



valores propios < 0

(i) repulsor (ii) atractor (iii) punto silla. → valores propios de signo distinto.

Valores positivos



Ejercicio 3

Decidir si la siguiente afirmación es verdadera o falsa:

Si $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ y $\mathbf{v}(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ resuelven la ecuación

$$\mathbf{y}'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y}(t)$$

con $\boxed{\mathbf{u}(0) \perp \mathbf{v}(0)}$ y $\boxed{\mathbf{u}(1) \perp \mathbf{v}(1)}$, entonces $\mathbf{u}(t) \perp \mathbf{v}(t)$ para todo t.

Respuesta:

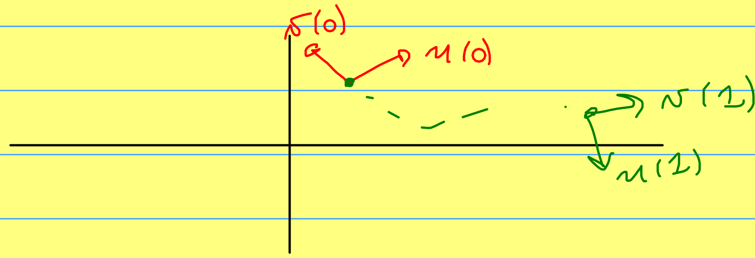
A) Verdadero

B) Falso

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$u' = Au$

$v' = Av$



Resolver el siguiente sistema reescribiendolo en forma matricial y usando valores y vectores propios:

$$\begin{cases} \dot{x} = 2x - y + t \\ \dot{y} = 3x - 2y - 1 \\ x(0) = 0, \; y(0) = 0 \end{cases}$$

Sugerencia: buscar soluciones de la forma

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_h \\ y_h \end{pmatrix} + \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

en donde $\begin{pmatrix} x_h \\ y_h \end{pmatrix}$ es solución de la ecuación homogénea y $\begin{pmatrix} x_p \\ y_p \end{pmatrix}$ es una solución particular de la forma $\begin{pmatrix} At + B \\ Ct + D \end{pmatrix}$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t \\ -1 \end{pmatrix}$$

Ecuación homogenea

$$\begin{pmatrix} At + B \\ Ct + D \end{pmatrix} = \begin{pmatrix} x_p(t) \\ y_p(t) \end{pmatrix}$$

$$A = \dot{x}_p = 2(At + B) - (Ct + D) + t$$

$$C = \dot{y}_p = 3(At + B) - 2(Ct + D) - 1$$

Matriz Fundamental

Si D es una matriz diagonal, $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

$\dot{x} = Dx \quad \rightsquigarrow \quad x(t) = x(0)\, \boxed{e^{Dt}}$

Def: $e^{Dt} = \sum_{k=0}^{\infty} \frac{(Dt)^k}{k!}$, entendiendo $(Dt)^0 = Id$

• $(Dt)^k = D^k t^k$

$D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

$D^k = \operatorname{diag}(\lambda_1^k, \dots, \lambda_n^k)$

$$e^{Dt} = \underbrace{\operatorname{diag}(1, \dots, 1)}_{Id} + \sum_{k=1}^{+\infty} \frac{\operatorname{diag}(\lambda_1^k t^k, \dots, \lambda_n^k t^k)}{k!}$$

Entrada $(D^t)_{ii}$ $= 1 + \sum_{k=1}^{+\infty} \frac{\lambda_i^k t^k}{k!} = e^{\lambda_i t}$ Taylor

$$e^{Dt} = \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 \dots & 0 \\ & \vdots & & \\ 0 & \dots & 0 & e^{\lambda_n t} \end{pmatrix}$$

$$x(t) = x(0)\, e^{Dt} = \left(x_1(0)\, e^{\lambda_1 t}, \dots, x_n(0)\, e^{\lambda_n t}\right)$$

¿Qué pasa cuando cambiamos la base?

$$e^{At} = e^{P^{-1} B P t}$$

Af: $(P^{-1} B P)^K = P^{-1} B^K P$

Demo. $K=1$ ✓

$K=n$: Sabemos que $(P^{-1}BP)^n = P^{-1} B^n P$

queremos ver que $(P^{-1}BP)^{n+1} = P^{-1}B^{n+1}P$

$$(P^{-1}BP)^{n+1} = (P^{-1}BP)^n (P^{-1}BP)$$

$$= P^{-1}B^n \underbrace{P\, P^{-1}}_{Id} BP = P^{-1}B^{n+1}P$$

$$e^{P^{-1}BP} = \sum_{K=0}^{+\infty} \frac{(P^{-1}BP)^K}{K!} = \sum_{K=0}^{+\infty} \frac{P^{-1}B^K P}{K!} = P^{-1} \underbrace{\left(\sum_{K=0}^{+\infty} \frac{B^K}{K!}\right)}_{e^B} P$$

$$At = P^{-1} B P t \to e^{At} = P^{-1} e^{Bt} P$$

1) $A = \begin{pmatrix} 4 & -6 \\ 2 & -4 \end{pmatrix}$ $\qquad e^A$

B: $(4-\lambda)(-4-\lambda) + 12 = 0$

$\lambda^2 - 16 + 12 = 0 \to \lambda^2 - 4 = 0 \to B = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$\lambda = \pm 2$

$$e^B = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-2} \end{pmatrix} \to B \text{ diagonal}$$

$$e^A = P^{-1} e^B P$$

P: v1: 1) $A = \begin{pmatrix} 4 & -6 \\ 2 & -4 \end{pmatrix}$ $\to A - 2I = \begin{pmatrix} 2 & -6 \\ 2 & -6 \end{pmatrix}$

$\lambda = 2$

$2x - 6y = 0 \to \boxed{x = 3y}$

$v1 = (3, 1)$

v2: $\lambda = -2$: $A + 2I = \begin{pmatrix} 6 & -6 \\ 4 & -4 \end{pmatrix}$

$6x - 6y = 0 \to \boxed{x = y} \to v_2 = (1, 1)$

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

$$e^A = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^{-2} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

¿Cómo se e^{J} cuando J es de Jordan?

$$J = \begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 \\ 1 & \lambda & 0 & 0 & \cdots & 0 \\ 0 & 1 & \lambda & 0 & \cdots & \\ \vdots & & \vdots & & & \vdots \\ \cdots & & & & 1 & \lambda \end{pmatrix}$$

Ej: $J = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$

$$\boxed{e^{J} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1^2}{2!} & 1 & 1 \end{pmatrix}}$$

e^{Jt}

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ \frac{t^2}{2!} & t & 1 & 0 \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 \end{pmatrix}$$

$A = P^{-1} J P \rightsquigarrow e^{At} = P^{-1} e^{Jt} P$

10. (a) Probar que si $\vec{A} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ entonces $e^{At} = e^{\alpha t} \boxed{\begin{pmatrix} \cos(\beta t) & \operatorname{sen}(\beta t) \\ -\operatorname{sen}(\beta t) & \cos(\beta t) \end{pmatrix}}$.

Nota: Para probar esta parte vamos a admitir el siguiente resultado:
Si A y B son dos matrices tales que $\underline{A.B = B.A}$ entonces $e^{(A+B)t} = e^{At}e^{Bt}$.

$$A = \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}}_{B} + \underbrace{\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}_{C}$$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha\beta \\ -\alpha\beta & 0 \end{pmatrix} \overset{\text{ver}}{=} \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

$$e^{At} = e^{Bt} e^{Ct}$$

$\to Bt = \text{diag}(\alpha t, \alpha t) \to e^{Bt} = \text{diag}(e^{\alpha t}, e^{\alpha t}) = e^{\alpha t} Id$

$$e^{At} = e^{\alpha t} \, I \, e^{Ct} = e^{\alpha t} e^{Ct}$$

Af: Si $C = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \Rightarrow e^{Ct} = \begin{pmatrix} \cos(\beta t) & \operatorname{sen}(\beta t) \\ -\operatorname{sen}(\beta t) & \cos(\beta t) \end{pmatrix}$

Sugerencia, i) Verificar que las potencias de C se "repite" despues de 4 potencias.

ii) Recordar los Taylor's de $\cos(\beta t)$ y $\operatorname{sen}(\beta t)$

2×2

$$A = P^{-1} B P \to B : \begin{cases} \to \text{Diagonal} \\ \to \text{Jordan} \\ \to \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \end{cases}$$

3×3: