(k) $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$

$$\dot{x} = A x$$

$$\begin{cases} x' = -3x + 2y \\ y' = -x - y \end{cases}$$

1) $p(\lambda) = (-3-\lambda)(-1-\lambda) + 2$
$= \lambda^2 + 4\lambda + 3 + 2$
$= \lambda^2 + 4\lambda + 5$

$\sqrt{2 \cdot 6} = \sqrt{2}\sqrt{6}$

$\sqrt{-1 \cdot 4} = \sqrt{-1}\sqrt{4} = i\,2$, $\sqrt{-1} = i$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm \frac{\sqrt{-4}}{2}$$

$$= -2 \pm i\frac{2}{2} \rightarrow -2 \pm i$$

$p(\lambda) = (-2+i)(-2-i)$

Los valores propios son complejos.

$\cdot\ A = PBP^{-1} \rightsquigarrow B = \begin{pmatrix} -2+i & 0 \\ 0 & -2-i \end{pmatrix}$

2) $P = (v_1 | v_2)$ donde v_1 es el vec. prop. asociado a $\lambda_1 = -2+i$
v_2 " " " " " $\lambda_2 = -2-i$

• v_1: $v_1 \in \mathrm{Ker}(A - \lambda_1 I)$

$A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$

$\lambda_1 = -2+i$
$-\lambda_1 = 2-i$
$\rightsquigarrow A - \lambda_1 I = \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}$

Queremos

$$A-\lambda_1 I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-1-i)x + 2y \\ -x + (1-i)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{i^2=-1}$

$$\begin{cases} (-1-i)x + 2y = 0 & (i) \\ -x + (1-i)y = 0 & (ii) \end{cases}$$

De (ii): $x = (1-i)y$, sustituyendo en (i):

$$(-1-i)(1-i)y + 2y = 0$$

$$[\overbrace{(-1-i)(1-i)}^{-2} + 2]\,y = 0$$

$\underbrace{-1}_{-b}\ \underbrace{-i}_{a}\quad \underbrace{1}_{b}\ \underbrace{-i}_{a}$

$\to y = 0$

$\to (-1-i)(1-i) + 2 = 0$

y está libre

$$(a-b)(a+b) = a^2 - b^2$$

$$\begin{aligned}(-1-i)(1-i) &= (-i)^2 - 1^2 \\ &= (-1\cdot i)^2 - 1 \\ &= (-1)^2 i^2 - 1 \\ &= i^2 - 1 = -2\end{aligned}$$

$\to i^2 = -1$

$$\overbrace{(-2+2)}^{0}y = 0 \to 0 = 0 \to$$ La variable $y \in \mathbb{C}$ está libre.

Podemos elegir $y=1$, de donde $x = 1-i$

$$\boxed{v_1 = (1-i, 1)}$$

$\underline{v_2}$: $\quad V_2 \in Ker(A - \lambda_2 I)$

$$A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$$

$$\left.\begin{array}{l} \lambda_2 = -2 - i \\ -\lambda_2 = 2 + i \end{array}\right\} A - \lambda_2 I = \begin{pmatrix} -1+i & 2 \\ -1 & 1+i \end{pmatrix}$$

$$\left\{\begin{array}{lr} (-1+i)x + 2y = 0 & (i) \\ -x + (1+i)y = 0 & (ii) \end{array}\right.$$

De (ii): $\quad x = (1+i)y$, sustituyendo en (i):

$$(-1+i)(1+i)y + 2y = 0$$

$$[\underbrace{(-1+i)(1+i)} + 2]\, y = 0 \to y = 0$$

$\to (-1+i)(1+i) + 2 = 0$

$y \in \mathbb{C}$ libre.

$$(-1+i)(1+i) = -1^2 + i^2 = -1 + i^2 = -1 - 1 = -2$$

$i^2 = -1$

$$(-a+b)(a+b) = -a^2 + b^2$$

$(-2+2)y = 0 \to 0 = 0 \to y$ libre.

Si $y = 1 \to x = 1+i \Rightarrow \boxed{v_2 = (1+i, 1)}$

$\to$ conjugado.

$V_2 = \overline{V_1}$

Donde el conjugado de un vector, es conjugar las entradas.

Entonces $P = \begin{pmatrix} 1-i & 1+i \\ 1 & 1 \end{pmatrix}$

$$B = \begin{pmatrix} -2+i & 0 \\ 0 & -2-i \end{pmatrix}$$

$$\boxed{A = PBP^{-1}}$$

$$\begin{cases} u' = (-2+\overset{1}{i})\,u \\ v' = (-2-i)\,v \end{cases}$$

$-1 \cdot i$

De la teoría sabemos que la solución a este nuevo sistema es:

$$u(t) = u_0\left(e^{-2t}\cos(1t) + i\,e^{-2t}\operatorname{sen}(1t)\right)$$
$$v(t) = v_0\left(e^{-2t}\cos(-1t) + i\,e^{-2t}\operatorname{sen}(-1t)\right)$$

Obs: $u' = (-2+i)u \to u(t) = e^{(-2+i)t} = Ae^{-2t}\,e^{it}$ $\to \cos(t) + i\operatorname{sen}(t)$

$$\frac{u'}{u} = 2+i \to \ln(u) = (2+i)t$$

Si $Y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$, entonces $X(t) = P\,Y(t)$

$$Y(t) = \begin{pmatrix} u_0\left(e^{-2t}\cos(t) + e^{-2t}\operatorname{sen}(t)\right) \\ v_0\left(e^{-2t}\underbrace{\cos(-t)}_{\cos(t)} + e^{-2t}\underbrace{\operatorname{sen}(-t)}_{-\operatorname{sen}(t)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} u_0\, e^{-2t}\left(\cos(t) + \operatorname{sen}(t)\right) \\ v_0\, e^{-2t}\left(\cos(t) - \operatorname{sen}(t)\right) \end{pmatrix}$$

Ojo: Saque la unidad imaginaria.

Por último:

$$X(t) = \begin{pmatrix} 1-i & 1+i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_0 e^{-2t}(\cos(t) + \cancel{i}\,\text{sen}(t)) \\ v_0 e^{-2t}(\cos(t) - \cancel{i}\,\text{sen}(t)) \end{pmatrix}$$

$$-i\,(u_0 e^{-2t}(\cos(t) + \text{sen}(t))) + i\, v_0 (e^{-2t}(\cos(t) - \text{sen}(t)))$$

$$\text{"} \quad i e^{-2t}[-u_0 \cos(t) - u_0 \text{sen}(t) + v_0 \cos(t) - v_0 \text{sen}(t)]$$

$$i e^{-2t}[-u_0(\cos t + \text{sen}\, t) + v_0(\cos t - \text{sen}\, t)]$$

<u>Obs</u>: La solución $X(t)$ tiene que ser real.

8. Escribir las siguientes ecuaciones diferenciales lineales a coeficientes constantes en la forma $x' = Ax$, y resolver:

(a) $x'' + x' - 2x = 0$ (b) $x'' + x = 0$ (c) $x''' - 2x'' - x' + 2x = 0$

a) $x'' + x' - 2x = 0 \rightsquigarrow \begin{cases} x' = - \\ y' = - \end{cases} \to \dot{X} = AX$

Definimos $y = x' \rightsquigarrow y' = x'' = -x' + 2x$

$\begin{cases} x' = y \\ y' = -y + 2x \end{cases} \rightsquigarrow \boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}}$

$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

1) $p(\lambda) = -\lambda(-1-\lambda) - 2$

$= \lambda^2 + \lambda - 2 \to \lambda = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2}$ $\quad \sqrt{9} = 3$

$\lambda_1 = 1 \qquad \lambda_2 = -2$

$A = PDP^{-1}$ con $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

2) $v_1 \in \ker(A - \lambda_1 I)$:

$\left.\begin{array}{l} A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \\ \lambda_1 = 1 \end{array}\right\} \quad A - \lambda_1 I = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$

$$\begin{cases} \boxed{-x + y = 0} & (i) \\ 2x - 2y = 0 & (ii) \end{cases}$$

$(ii) = -2\,(i) \rightarrow -x + y = 0 \rightarrow \boxed{x = y}$

Entonces $\boxed{v_1 = (1,1)}$

$\underline{v_2} \in \mathrm{Ker}(A - \lambda_2 I)$

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \rightarrow A - \lambda_2 I = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

$\lambda_2 = -2$

$$\begin{cases} 2x + y = 0 \\ 2x + y = 0 \end{cases} \rightarrow \boxed{y = -2x}$$

$$\boxed{v_2 = (1,-2)}$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow X(t) = P\,Y(t)$$

con $Y(t)$ solución de $\dot{Y} = D\,Y$

$$\begin{cases} u' = u \\ v' = -2v \end{cases} \rightarrow \begin{array}{l} u(t) = u_0\, e^{t} \\ v(t) = v_0\, e^{-2t} \end{array}$$

$$Y(t) = \begin{pmatrix} u_0\, e^{t} \\ v_0\, e^{-2t} \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

$$X(t) = P\,Y(t) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} u_0 e^{t} \\ v_0 e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} u_0 e^{t} + v_0 e^{-2t} \\ u_0 e^{t} - 2 v_0 e^{-2t} \end{pmatrix}$$

$$\begin{cases} \dot{X} = AX \\ X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{cases} \rightarrow X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} u_0 + v_0 \\ u_0 - 2v_0 \end{pmatrix}$$

$$\begin{cases} u_0 + v_0 = x_0 \\ u_0 - 2v_0 = y_0 \end{cases}$$

$$\rightarrow u_0 = x_0 - v_0 \rightarrow u_0 = x_0 - \frac{x_0 - y_0}{3} = \frac{2x_0 - y_0}{3}$$

$$\rightarrow x_0 - v_0 - 2v_0 = y_0$$

$$-3v_0 = y_0 - x_0$$

$$\boxed{v_0 = \frac{x_0 - y_0}{3}}$$

$x(t)$

$$X(t) = \begin{pmatrix} \frac{2x_0 - y_0}{3} e^{t} + \frac{x_0 - y_0}{3} e^{-2t} \\ \frac{2x_0 - y_0}{3} e^{t} - 2\left(\frac{x_0 - y_0}{3}\right) e^{-2t} \end{pmatrix}$$

$y(t) = x'(t)$

→ Esta es la solución que buscamos:

$$\left.\begin{array}{l} x'' + x' - 2x = 0 \\ \lambda_1 = 1 \\ \lambda_2 = -2 \end{array}\right\} \quad x(t) = C_1 e^{t} + C_2 e^{-2t}$$

$$x'''' = -3x''' - 2x'' + 5x' + x$$

$$\underbrace{x'''}_{w} \quad \underbrace{x''}_{z} \quad \underbrace{x'}_{y}$$

$$x'''' + 3x''' + 2x'' - 5x' - x = 0$$

$$y = x', \quad z = x'', \quad w = x'''$$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \to X' = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} x' \\ x'' \\ x''' \\ x'''' \end{pmatrix} = \begin{pmatrix} y \\ z \\ w \\ -3w - 2z + 5y + x \end{pmatrix}$$

$$\begin{pmatrix} y \\ z \\ w \\ w' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = X' = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 5 & -2 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$\begin{cases} y = y \\ z = z \\ w = w \\ w' = x + 5y - 2z + w \end{cases}$$