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4. Encontrar la solución general del sistema lineal

$$\begin{cases} x' = x \\ y' = \alpha y \end{cases}$$

donde $\alpha$ es una constante real. Hacer un esquema del diagrama de fase para los valores $\alpha = -1, 0, 1$.

**Observar**: la estructura cualitativa del diagrama de fase es la misma para todos los valores $\alpha < 0$, así como para todos los $\alpha > 0$, sin embargo cambia en el parámetro $\alpha = 0$.

$\alpha = 1$: $\begin{cases} x' = x \\ y' = y \end{cases} \to \begin{array}{l} x(t) = x_0 e^{t} \\ y(t) = y_0 e^{\alpha t} \end{array}$

$x_0 \neq 0$.

$e^t = \dfrac{x(t)}{x_0} \to t = \ln\left(\dfrac{x(t)}{x_0}\right)$

$y(t) = y_0 \, e^{\alpha \ln\left(\frac{x(t)}{x_0}\right)} = \overbrace{\dfrac{y_0}{x_0^{\alpha}}}^{m} x(t) \to \boxed{y = m x}$

$\alpha = -1$: $\begin{cases} x' = x \\ y' = -y \end{cases} \to \begin{array}{l} x(t) = x_0 e^{t} \\ y(t) = y_0 e^{-t} \end{array}$

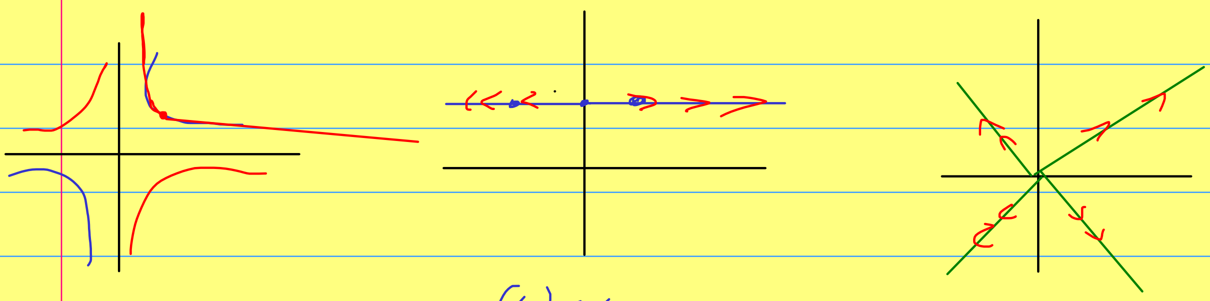
$t = \ln\left(\dfrac{x}{x_0}\right)$

$\to y = y_0 e^{-\ln\left(\frac{x}{x_0}\right)} = y_0 \cdot \dfrac{1}{e^{\alpha \ln\left(\frac{x}{x_0}\right)}} = y_0 \cdot \dfrac{1}{\frac{x}{x_0}} = \dfrac{y_0 x_0}{x}$

$\boxed{y = \dfrac{y_0 x_0}{x}}$ $\to$ $\boxed{xy = y_0 x_0}$

$\alpha = 0$: $\begin{cases} x' = x \\ y' = 0 \end{cases} \to \begin{array}{l} x(t) = x_0 e^{t} \\ y(t) = y_0 \end{array}$

$\alpha < 0$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha > 0$ $\qquad$ $y = \frac{y_0}{x_0} x$



$y(t) = y_0$
$x(t) = x_0 e^t$

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(e) $\begin{cases} x' = x \\ y' = y \\ z' = x \end{cases}$

Una forma:

$x' = x \to x(t) = x_0 e^t$

$y' = y \to y(t) = y_0 e^t$

$z' = x \qquad z' = x_0 e^t \; = f(t)$

$$\int \overbrace{z' \, dt}^{dz} = \int x_0 e^t \, dt$$

$$\underset{z(t)}{\shortparallel} \qquad \underset{x_0 e^t + K}{\shortparallel}$$

$\boxed{z(t) = x_0 e^t + K}$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (1-\lambda)(1-\lambda)(-\lambda) = -(1-\lambda)^2 \lambda$$

Valores propios: $\lambda = 0$

$\lambda = 1$ con $ma(1) = 2$

• Veamos si $m_g(1) = 2$, en cuyo caso $A$ es diagonalizable.

$m_g(1) = \dim(\ker(A - I))$

$$A - I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$\underline{\ker(A-I)}$ : $(A - I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} 0 = 0 \\ 0 = 0 \\ x - z = 0 \end{cases} \rightarrow \boxed{x = z}$$

$\ker(A - I) = \{(x,y,z) \in \mathbb{R}^3 : x = z\}$

$\dim(\ker(A - I)) = 2 \Rightarrow m_g(1) = m_a(1)$ ✓

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P = (\, v_1 \mid v_2 \mid v_0 \,)$$

$v_1 = (1, 0, 1)$ , $v_2 = (1, 1, 1)$

$\underline{v_0}$: $v_0 \in \ker(A - 0I) = \ker(A)$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \begin{cases} x = 0 \\ y = 0 \\ x = 0 \end{cases} \rightsquigarrow z \text{ está libre}$$

$\ker(A) = \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}$

$v_0 = (0, 0, 1)$

Entonces $P = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$
$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

- Tenemos hallar las soluciones a $y' = Dy$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

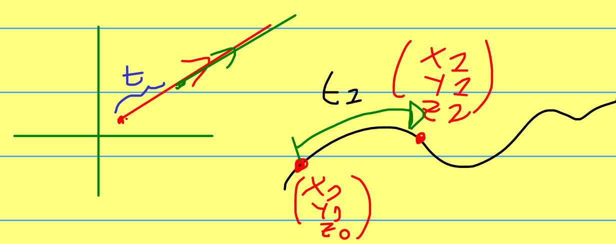
$$y(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$$

$$\begin{cases} u' = u & \to u(t) = u_0 e^t \\ v' = v & \to v(t) = v_0 e^t \\ w' = 0 & \to w(t) = w_0 \end{cases}$$

$$y(t) = \begin{pmatrix} u_0 e^t \\ v_0 e^t \\ w_0 \end{pmatrix}$$

- $$x(t) = P\,y(t) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} y(t) = \begin{pmatrix} u_0 e^t + v_0 e^t \\ v_0 e^t \\ u_0 e^t + v_0 e^t + w_0 \end{pmatrix}$$

$$\begin{cases} x' = Ax \\ \boxed{x(0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}} \end{cases}$$




$$\begin{pmatrix} x_1 \\ y_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + x(t_1)$$

<u>Objetivo</u>: Hallar $u_0, v_0$ y $w_0$ en función de $x_0, y_0, z_0$

$$X(0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\begin{pmatrix} u_0 + v_0 \\ v_0 \\ u_0 + v_0 + w_0 \end{pmatrix} \rightsquigarrow \begin{cases} u_0 + v_0 = x_0 \\ \boxed{v_0 = y_0} \\ u_0 + v_0 + w_0 = z_0 \end{cases}$$

$$\begin{cases} u_0 + y_0 = x_0 \to \boxed{u_0 = x_0 - y_0} \\ u_0 + y_0 + w_0 = z_0 \to x_0 \cancel{- y_0} \cancel{+ y_0} + w_0 = z_0 \end{cases}$$

$$x_0 e^t \cancel{- y_0 e^t} \quad \cancel{+ y_0 e^t} \qquad \boxed{w_0 = z_0 - x_0}$$

$$X(t) = \begin{pmatrix} (x_0 - y_0) e^t + y_0 e^t \\ y_0 e^t \\ (x_0 - y_0) e^t + y_0 e^t + z_0 - x_0 \end{pmatrix} = \begin{pmatrix} x_0 e^t \\ y_0 e^t \\ x_0 e^t + z_0 - x_0 \end{pmatrix} \quad K$$

$$x_0 e^t$$

# Valores propios complejos

(k) $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$

- $p(\lambda) = \det(A - \lambda I) = (-3-\lambda)(-1-\lambda) + 2$

$= \lambda^2 + 4\lambda + 3 + 2$

$= \lambda^2 + 4\lambda + 5$

$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \overbrace{\sqrt{-4}}^{i2}}{2} = -2 \pm i$

$\sqrt{ab} = \sqrt{a}\sqrt{b}$

$\sqrt{-4} = \underbrace{\sqrt{-1}}_{i}\underbrace{\sqrt{4}}_{2}$

Valores propios

$\lambda_1 = -2 + i$

$\lambda_2 = \overline{\lambda_1}$

$PAP^{-1} = B$

Cuando los valores propios son complejos.

Del teórico

$a = Re(\lambda)$, $b = Im(\lambda) \rightarrow a = -2$, $b = 1$

$B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \rightarrow B = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$

- ¿Cómo hallamos P?

Sea v el vector propio asociado a $\lambda_1$

$v = (z, w)$, $z, w \in \mathbb{C}$

$z = c + i d \quad \text{Re}(v) \quad \text{Im}(v)$

$w = e + i f$

$\to \text{Re}(v) := (c, e)$

$\text{Im}(v) := (d, f)$

Ej: $v = (1 - 3i, 4i) \to \text{Re}(v) = (1, 0)$

$\text{Im}(v) = (-3, 4)$

$\to P = (\text{Re}(v) \mid \text{Im}(v))$

- $\lambda_1 = -2 + i. \to -\lambda_1 = 2 - i$

$\underline{v \in \text{Ker}(A - \lambda_1 I)}$

(k) $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$ $\to A - \lambda_1 I = \begin{pmatrix} -1 - i & 2 \\ -1 & 1 - i \end{pmatrix}$

$\boxed{i^2 = -1}$

$$\begin{cases} (-1 - i)x + 2y = 0 & (i) \\ -x + (1 - i)y = 0 & (ii) \end{cases}$$

De (ii): $x = (1 - i)y$, sustituyendo en (i):

$(-1 - i)(1 - i)y + 2y = 0$

$[(\underbrace{-1}_{-b} - \underbrace{i}_{a})(\underbrace{1}_{b} - \underbrace{i}_{a}) + 2]\, y = 0 \to y = 0$

$\to (-1 - i)(1 - i) + 2 = 0$

$y$ libre.

Obs: $(a + b)(a - b) = a^2 - b^2$

$(-1 - i)(1 - i) = (-i)^2 - 1^2$

$= (-i)^2 - 1$

$= (-1 \cdot i)^2 - 1$

$$= (-1)^2 i^2 - 1$$
$$= i^2 - 1$$
$$= -1 - 1 = -2$$

$$v \in \mathrm{Ker}(A - \lambda_1 I) = \{((1-i)y, y) : y \in \mathbb{C}\}$$

$$v = (1-i, 1) \rightsquigarrow \mathrm{Re}(v) = (1, 1)$$
$$\mathrm{Im}(v) = (-1, 0)$$

$$P = (\mathrm{Re}(v) \mid \mathrm{Im}(v)) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

• ¿Cómo son las soluciones a,

$$B = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \ ?$$

$\lambda = -2 + i$

$$Y(t) = \begin{pmatrix} r_0 e^{-2t} \cos(t - \theta_0) \\ r_0 e^{-2t} \operatorname{sen}(t - \theta_0) \end{pmatrix}$$

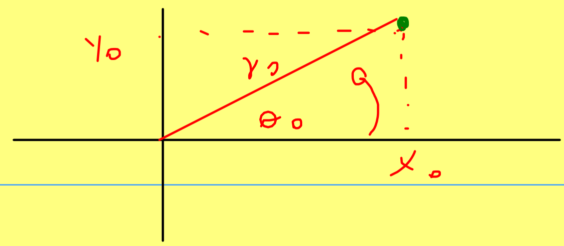
En general:

$$Y(t) = \begin{pmatrix} r_0 e^{\mathrm{Re}(\lambda) t} \cos(\mathrm{Im}(\lambda) t - \theta_0) \\ r_0 e^{\mathrm{Re}(\lambda) t} \operatorname{sen}(\mathrm{Im}(\lambda) t - \theta_0) \end{pmatrix}$$

$$X(t) = P\,Y(t)$$

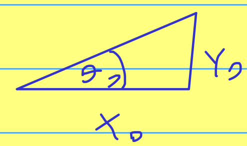
$$= r_0 e^{-2t} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(t - \theta_0) \\ \operatorname{sen}(t - \theta_0) \end{pmatrix} = r_0 e^{-2t} \begin{pmatrix} \cos(t-\theta_0) - \operatorname{sen}(t - \theta_0) \\ \cos(t - \theta_0) \end{pmatrix}$$

$$X(t) = \begin{pmatrix} r_0 e^{-2t} (\cos(t - \theta_0) - \operatorname{sen}(t - \theta_0)) \\ r_0 e^{-2t} \cos(t - \theta_0) \end{pmatrix}$$




$r_0 = \|(x_0, y_0)\| = \sqrt{x_0^2 + y_0^2}$

$\theta_0 = \operatorname{arctg}\left(\frac{y_0}{x_0}\right)$

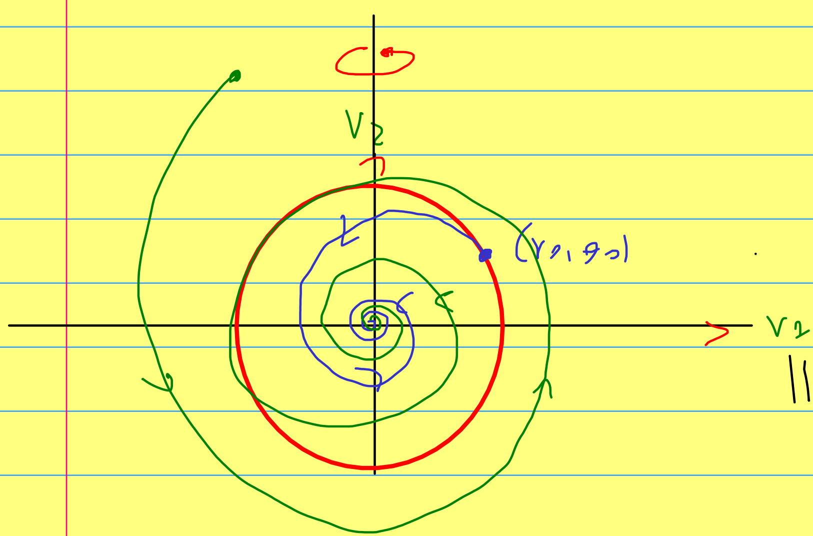



$$x(t) = \begin{pmatrix} \sqrt{x_0^2+y_0^2}\; e^{-2t}\left[\cos\left(t - \operatorname{arctg}\left(\frac{y_0}{x_0}\right)\right) - \operatorname{sen}\left(t - \operatorname{arctg}\left(\frac{y_0}{x_0}\right)\right)\right] \\ \sqrt{x_0^2+y_0^2}\; e^{-2t} \cos\left(t - \operatorname{arctg}\left(\frac{y_0}{x_0}\right)\right) \end{pmatrix}$$

$v = (1 - i, 1)$ , $\{Re(v), Im(v)\}$ es una base $\mathbb{R}^2$

$\{(1,1), (-1,0)\}$

1º) Dibujar el diagrama de$^{v_1}$ fase$^{v_2}$ en la base dada P




$$y(t) = \begin{pmatrix} r_0\, e^{-2t} \cos(t - \theta_0) \\ r_0\, e^{-2t} \operatorname{sen}(t - \theta_0) \end{pmatrix} = r_0 e^{-2t} \begin{pmatrix} \cos(t-\theta_0) \\ \operatorname{sen}(t-\theta_0) \end{pmatrix}$$

$$\|y(t)\| = \sqrt{(r_0 e^{-2t})^2 \underbrace{(\cos^2 + \operatorname{sen}^2)}_{1}}$$

$$= r_0 e^{-2t} \xrightarrow[t \to +\infty]{} 0$$

Moraleja: $\lambda = a + ib$ , $b \neq 0$

$a = 0 \rightarrow$ Da círculos centrados en el origen.

$a < 0 \rightarrow$ Espirales que "caen" al origen.

$a > 0 \rightarrow$ Espirales que se alejan del origen.

$b < 0 \to$ Sentido horario

$b > 0 \to$ Sentido horario