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· $f_n(x) = \sqrt{n}\, x\,(1-x^2)^n$

$x \in [0,1]$

x_0 fijo: $\lim_n |f_n(x_0) - g(x_0)| = 0$

Si $x_0 \in [0,1] \Rightarrow 0 \leq 1 - x_0^2 \leq 1$

$f(x) = \sqrt{n}\, x\,(1-x^2)^n$

$f'(x) = \sqrt{n}\,(1-x^2)^n + n\sqrt{n}\, x\,(1-x^2)^{n-1}(-2x)$

$= \sqrt{n}\left[(1-x^2)^n - 2n x^2 (1-x^2)^{n-1}\right]$

$= \sqrt{n}\,(1-x^2)^{n-1}\left[\underbrace{(1-x^2) - 2n x^2}\right]$

$-(2n+1)x^2 + 1 = 0$

$x^2 = \dfrac{1}{2n+1}$

$x = \pm\sqrt{\dfrac{1}{2n+1}}$

$-\sqrt{\dfrac{1}{2n+1}} \notin [0,1]$

El máximo se da en $x_0 = \sqrt{\dfrac{1}{2n+1}}$

$$f(x) = \sqrt{n}\; x\,(1 - x^2)^n$$

$$f(x_0) = \sqrt{n}\,\sqrt{\frac{1}{2n+1}}\left(1 - \frac{1}{2n+1}\right)^n$$

$$= \sqrt{\frac{n}{2n+1}}\left(\frac{2n}{2n+1}\right)^n$$

$$= \frac{\sqrt{n}}{\sqrt{2n+1}}\,\frac{(2n)^n}{(2n+1)^n} = \frac{\sqrt{n}\,(2n)^n}{(2n+1)^{n+\frac{1}{2}}}$$

$$= \frac{\overset{n^{1/2}}{\sqrt{n}}\; 2^n \; n^n}{(2n+1)^{n+\frac{1}{2}}} = 2^n \frac{n^{n+\frac{1}{2}}}{(2n+1)^{n+1/2}}$$

$$= 2^n \left(\frac{n}{2n+1}\right)^{n+\frac{1}{2}}$$

$$\frac{n}{2n+1} < \frac{n}{2n} = \frac{1}{2} \qquad < \underbrace{2^n \left(\frac{1}{2}\right)^n}_{\left(\frac{2}{2}\right)^n = 1} \sqrt{1/2} = \sqrt{1/2} \quad \forall n$$

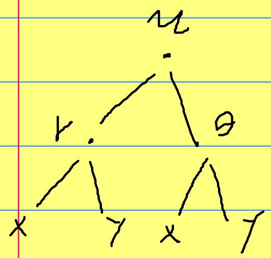
Ejercicio 4 (1 punto)

Considerar la ecuación de Laplace en el disco unidad

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1$$

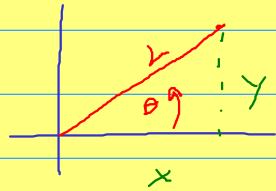
1. En coordenadas polares $x = r\cos\theta$, $y = r\sin\theta$ la ecuación es:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$



$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$



$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\boxed{\overset{\cos\theta}{\frac{\partial r}{\partial x}}} + \frac{\partial}{\partial \theta}\boxed{\frac{\partial \theta}{\partial x}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\boxed{\overset{\operatorname{sen}\theta}{\frac{\partial r}{\partial y}}} + \frac{\partial}{\partial \theta}\boxed{\frac{\partial \theta}{\partial y}}$$

$$r = \sqrt{x^2+y^2} \to \frac{\partial r}{\partial x} = \frac{\overset{r\cos\theta}{x}}{\underbrace{\sqrt{x^2+y^2}}_{r}} = \cos\theta \to \boxed{\frac{\partial r}{\partial x} = \cos\theta}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \to \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{0\cdot x - y}{x^2}\right)$$

$$= \frac{-y}{x^2\left(1+\frac{y^2}{x^2}\right)} = -\frac{\overset{r\operatorname{sen}\theta}{y}}{\underbrace{x^2+y^2}_{r^2}}$$

$$\boxed{\frac{\partial \theta}{\partial x} = -\frac{\operatorname{sen}\theta}{r}}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{x+\frac{y^2}{x}} = \frac{1}{\frac{x^2+y^2}{x}} = \frac{\overset{r\cos\theta}{x}}{\underbrace{x^2+y^2}_{r^2}}$$

$$\boxed{\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}}$$

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \operatorname{sen}\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r} \frac{\partial}{\partial \theta}\right)^2 + \left(\operatorname{sen}\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\right)^2$$

$= \cdot \quad \cdot \quad - \; -$

$= \cdot \; - \quad \cdot$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

2. Si buscamos una solución de la forma $u(r,\theta) = R(r)\Theta(\theta)$, las ecuaciones diferenciales que deben satisfacer $R(r)$ y $\Theta(\theta)$ son:

$$u(r,\theta) = R(r)\,\Theta(\theta) \qquad \Delta u = 0$$

$$\underbrace{R''\Theta + \frac{1}{r}R'\Theta}_{\left(R'' + \frac{R'}{r}\right)\Theta} + \frac{1}{r^2}R\Theta'' = 0$$

$$\left(R'' + \frac{R'}{r}\right)\Theta = -\frac{1}{r^2} R\,\Theta''$$

$$\frac{r^2}{R}\left(R'' + \frac{R'}{r}\right) = -\frac{\Theta''}{\Theta}$$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Theta''}{\Theta} = K$$

$$\begin{cases} r^2 R'' + r R' = K R \\ \Theta'' = -K\,\Theta \end{cases}$$

K constante

3. Suponga que $u(r,\theta) = R(r)\Theta(\theta)$ es solución, entonces la fórmula general para $R(r)$ y $\Theta(\theta)$ es:

$$\lambda^2 = -K \qquad (K > 0)$$

$$\Theta(\theta) = A\cos(\sqrt{K}\,\theta) + B\,\mathrm{sen}(\sqrt{K}\,\theta)$$

$$\boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)}$$

$$r^2 R'' + r R' = K R \rightarrow \boxed{R(r) = r^\alpha}$$

$$r^2(\alpha(\alpha-2))\,r^{\alpha-2} + r\,\alpha\,r^{\alpha-2} = K r^\alpha$$

$$\alpha(\alpha-2)\,r^\alpha + \alpha\,r^\alpha = K r^\alpha$$

$$[\underbrace{\alpha(\alpha-2)}_{\alpha^2\,\alpha+\alpha} + \alpha]\,\cancel{r^\alpha} = K\cancel{r^\alpha}$$

$$\alpha^2 = K \rightarrow \alpha = \pm\sqrt{K} = \pm n$$

$$\rightarrow \boxed{R_n(r) = C_n r^n + D_n r^{-n}}$$

• $n=0$: $(K=0)$

$$r^2 R'' + r R' = 0$$

$$r^2 R'' = -r R'$$

$$\left(\ln(R')\right)' = \frac{R''}{R'} = -\frac{1}{r}$$

$$\left(\ln(R')\right)' = -\frac{1}{r}$$

$$\ln(R') = -\ln(r) + \beta \quad \text{cte}$$

$$\ln(R') + \ln(r) = \beta$$

$$\ln(r R') = \beta \rightarrow r R' = e^{\beta}$$

$$R' = \frac{1}{r} e^{\beta}$$

$$R = \int R' \, dr = \underbrace{e^{\beta}}_{A} \underbrace{\int \frac{1}{r} \, dr}_{\ln(r)} + \Gamma$$

Si $n=0$: $\boxed{R_0(r) = \Gamma_0 + A_0 \ln(r)}$

$$u(r,\theta) = \overbrace{\Gamma_0 + A_0 \ln(r)}^{A_0} + \sum_{n=1}^{+\infty} (A_n\cos(n\theta) + B_n\sin(n\theta))(C_n r^n + D_n r^{-n})$$

$$u(r,\theta) = \underset{\theta_0}{A_0}(\overline{\Gamma_0 + \Lambda_0 \ln(r)}^{\,k_0}) + \sum_{n=1}^{+\infty} \big(\underbrace{A_n \cos(n\theta) + B_n \sin(n\theta)\big)\big(\cancel{C_n r^n} + D_n}_{\tilde{A}_n,\ \tilde{B}_n} r^{-n}\big)$$

$\mathring{D} = \{(x,y) = x^2 + y^2 < 1\}$

D es compacto
u es continua $\Big\} \Rightarrow$ Está acotada $\Rightarrow C_n = 0$

$$\boxed{u(r,\theta) = A + B\ln(r) + \sum_{n=1}^{+\infty} (\tilde{A}_n \cos(n\theta) + \tilde{B}_n \sin(n\theta))\, r^{-n}}$$