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9. Sea $\{f_n\}$ una sucesión de funciones ($f_n : \mathbb{R} \to \mathbb{R}$) tal que

$$\boxed{f_n''(x) = f_{n+2}(x)(n+2)(n+1) - f_n(x) \quad \forall x \in \mathbb{R} \quad \forall n \in \mathbb{N}}$$

$$\text{con } \boxed{f_0(x) = e^x} \text{ y } \boxed{f_1(x) = 0}$$

a) Probar que $f_{2n+1}(x) = 0, \forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$ y que $f_{2n}(x) = 2^n e^x/(2n)!$ $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$.

Queremos ver que $f_{2n+1}(x) = 0 \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N} \cup \{0\}$

- Paso base: Si $n = 0 \Rightarrow 2n+1 = 1 \Rightarrow f_1(x) = 0$ ✓

- Paso inductivo: Supongamos que $\underbrace{\text{se cumple para } n \text{ genérico}}_{f_{2n+1}(x) = 0}$

Queremos ver que $\underbrace{\text{se cumple para } n+1}$

$\to f_{2(n+1)+1}(x) = 0$

$2(n+1)+1 = 2n+3$: Supongamos $\boxed{f_{2n+1}(x) = 0}$

Ver que $f_{2n+3}(x) = 0$

$$f_n''(x) = \underbrace{f_{n+2}(x)}_{f_{2n+3}(x)}(n+2)(n+1) - f_n(x) \quad \forall x \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

Cambio n por $2n+1$

$$f_{2n+3}(x) = \frac{\overset{0}{\boxed{f''_{2n+1}(x)}} + \overset{0}{\boxed{f_{2n+1}(x)}}}{(2n+3)\,(2n+2)} =$$

$f_{2n+1}(x) = 0 \quad \forall x$

$\hookrightarrow$ cte

$\Rightarrow$

$f''_{2n+1}(x) = 0 \quad \forall x$

$$f_n''(x) = f_{n+2}(x)(n+2)(n+1) - f_n(x) \quad \forall x \in \mathbb{R} \ \ \forall n \in \mathbb{N}$$

con $\boxed{f_0(x) = e^x}$ y $f_1(x) = 0$

Probar $f_{2n}(x) = 2^n e^x/(2n)!$

- Paso base $n=0$: $f_0(x) = \dfrac{2^0 e^x}{(2.0)!} = \dfrac{1\, e^x}{1} = e^x$ ✓

- Paso inductivo: Supongamos que $f_{2n}(x) = \dfrac{2^n e^x}{(2n)!}$

Queremos ver que $f_{2(n+1)}(x) = \dfrac{2^{n+1} e^x}{(2(n+1))!}$

$2(n+1) = 2n+2$

$$f_n''(x) = f_{n+2}(x)(n+2)(n+1) - f_n(x) \quad \forall x \in \mathbb{R} \ \ \forall n \in \mathbb{N}$$

Cambio n por $2n$

$$f_{2n+2}(x) = \frac{f''_{2n}(x) + f_{2n}(x)}{(2n+2)(2n+1)} = \frac{2\left(\frac{2^n e^x}{(2n)!}\right)}{(2n+2)(2n+1)}$$

$$\boxed{f_{2n}(x) = \frac{2^n e^x}{(2n)!} \Rightarrow f''_{2n}(x) = \frac{2^n e^x}{(2n)!}}$$

$$= \frac{2.2^n e^x}{(2n)!\,(2n+1)(2n+2)} = \frac{2^{n+1} e^x}{(2(n+1))!}$$

$2.2^n = 2^{n+1}$

$(2n)!\,(2n+1)(2n+2) = (2n+2)! = (2(n+1))!$

b) Se considera la ecuación

$$u_{tt}(x,t) = u_{xx}(x,t) + u(x,t) \text{ con } -L < x < L \text{ y } 0 < t < 1$$

1) Buscar soluciones de la forma

$$u(x,t) = \sum_{n=0}^{+\infty} f_n(x) t^n,$$

dando una expresión explícita para $f_n(x)$.

Para usar la parte (a), hay que ver si

$$\begin{cases} f_n''(x) = f_{n+2}(x)(n+2)(n+2) - f_n(x) \\ f_2(x) = 0 \\ f_0(x) = e^x \end{cases}$$

. Sabemos que $\partial_x u(x,t) = \sum_{n=0}^{+\infty} \partial_x f_n(x) t^n$ si y sólo si

$$\sum_{n=0}^{K} f_n(x) t^n \rightrightarrows u(x,t)$$ y además

$$\sum_{n=0}^{K} \partial_x f_n(x) t^n \rightrightarrows h(x,t) = \partial_x u(x,t)$$

Vamos a suponer que podemos derivar adentro de la serie, y luego de encontrar $u(x,t)$ tenemos que verificarlo.

$$u_{tt} = u_{xx} + u, \quad u(x,t) = \sum_{n=0}^{+\infty} f_n(x) t^n$$

$$\partial_x^2 u(x,t) = \sum_{n=0}^{+\infty} f_n''(x) t^n$$

$$\partial_t^2 u(x,t) = \sum_{n=0}^{+\infty} n(n-2) f_n(x) t^{n-2} = \sum_{n=2}^{+\infty} n(n-2) f_n(x) t^{n-2}$$

$$\begin{cases} f_n''(x) = f_{n+2}(x)(n+2)(n+2) - f_n(x) \\ f_0(x) = e^x \\ f_2(x) = 0 \end{cases} \Rightarrow \quad f_{2n}(x) = \frac{2^n e^x}{(2n)!} \qquad f_{2n+2}(x) = 0$$

$$u_{tt} = u_{xx} + u$$

$$\underbrace{\sum_{n=2}^{+\infty} n(n-2) f_n(x) t^{n-2}} = \underbrace{\sum_{n=0}^{+\infty} f_n''(x) t^n + \sum_{n=0}^{+\infty} f_n(x) t^n}$$

$$\sum_{n=0}^{+\infty} (n+2)(n+2) f_{n+2}(x) t^n = \sum_{n=0}^{+\infty} f_n''(x) t^n + f_n(x) t^n$$

$$\sum_{n=0}^{+\infty} f_n''(x) t^n - f_{n+2}(x)(n+2)(n+2) t^n + f_n(x) t^n = 0$$

$$\sum_{n=0}^{+\infty} \left[f_n''(x) - f_{n+2}(x)(n+2)(n+2) + f_n(x) \right] t^n = 0$$

• Supongamos: $f_n''(x) - f_{n+2}(x)(n+2)(n+2) + f_n(x) = 0$

Parte $\leftarrow$ $f_n''(x) = f_{n+2}(x)(n+2)(n+2) - f_n(x)$

• Supongamos: $f_0(x) = e^x$, $f_2(x) = 0$

$$f_{2n}(x) = \frac{2^n e^x}{(2n)!} \qquad f_{2n+2}(x) = 0$$

$$u(t,x) = \sum_{n=0}^{+\infty} f_n(x) t^n = \sum_{n=0}^{+\infty} f_{2n}(x) t^n = \sum_{n=0}^{+\infty} \frac{2^n e^x}{(2n)!} t^n$$

Una vez encontrado el candidato, tenemos que verificar todas las suposiciones: • $f_0(x) = e^x$, $f_2(x) = 0$

• $f_n''(x) = f_{2n}(x)(n+2)(n+2) - f_n(x)$

1) $\cdot \sum_{n=0}^{k} f_{2n}(x) t^n \rightrightarrows u(x,t)$

2) $\cdot \sum_{n=0}^{K} \partial_x f_{2n}(x) t^n \rightrightarrows h(x,t) = \partial_x u(x,t)$

3) $\cdot \sum_{n=0}^{K} \partial^2_x f_{2n}(x) t^n \rightrightarrows j(x,t) = \partial^2_x u(x,t)$

4) $\cdot \sum_{n=0}^{K} \partial_t \left(f_{2n}(x) t^n\right) \rightrightarrows \ell(x,t) = \partial_t u(x,t)$

5) $\cdot \sum_{n=0}^{K} \partial^2_t \left(f_{2n}(x) t^n\right) \rightrightarrows m(x,t) = \partial^2_t u(x,t)$

1) $S_K = \sum_{n=0}^{K} \underbrace{\frac{2^n e^x t^n}{(2n)!}}_{g_n(x)} \rightrightarrows u(x,t)$ ✓

Mayorante: Si $|g_n(x)| \leq A_n \in \mathbb{R}$ y además $\sum_{n=0}^{+\infty} A_n$ converge

entonces $S_K \rightrightarrows S_\infty$

$u_{tt}(x,t) = u_{xx}(x,t) + u(x,t)$ con $\boxed{-L < x < L \text{ y } 0 < t < 1}$

$$\left|\frac{2^n e^x t^n}{(2n)!}\right| = \left|\frac{2^n}{(2n)!}\right| |e^x| \, |t^n| \leq \frac{2^n}{(2n)!} e^L 1^n = \frac{2^n}{(2n)!} e^L$$

es decir $A_n = \frac{2^n}{(2n)!} e^L$.

Hay que ver que $e^L \sum_{n=0}^{+\infty} \frac{2^n}{(2n)!}$ converge

$$a_n = \frac{2^n}{(2n)!} \quad , \quad \frac{a_n}{a_{n+1}} = \frac{\cancel{2^n}}{\cancel{(2n)!}} \cdot \frac{(2(n+1))!}{2^{\cancel{n+1}}} = \frac{(2n+2)(2n+1)}{2} > 1 \checkmark$$

$$(2(n+1))! = (2n+2)! = (2n+2)(2n+1)\cancel{(2n)!}$$

2) $\sum_{n=0}^{\infty} \underbrace{\partial_x \left(\frac{2^n e^x}{(2n)!} t^n \right)}_{= \frac{2^n}{(2n)!} e^x t^n} \Rightarrow h(x,t)$

$$\left| \partial_x \left(\frac{2^n e^x}{(2n)!} t^n \right) \right| = \left| \frac{2^n e^x t^n}{(2n)!} \right| \leq \frac{2^n}{(2n)!} e^L$$

Lo mismo que antes

$\hookrightarrow \partial_x u(x,t) = u(x,t)$

3) Obs: $\partial_x^2 \cdot \left(\frac{2^n e^x}{(2n)!} t^n \right) = \frac{2^n e^x}{(2n)!} t^n$

$\hookrightarrow \partial_x^2 u(x,t) = u(x,t)$

4) $\sum_{n=0}^{k} \partial_t \left(\frac{2^n e^x}{(2n)!} t^n \right) \rightrightarrows \ell(x,t)$

$$\left| \partial_t \left(\frac{2^n e^x}{(2n)!} t^n \right) \right| = \left| \frac{2^n e^x n t^{n-1}}{(2n)!} \right| \leq \overbrace{\frac{2^n n}{(2n)!} e^L}^{A_n}$$

Alcanza con ver que $\sum_{n=0}^{+\infty} \overbrace{\frac{2^n n}{(2n)!}}^{a_n}$ converge

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)}{\underbrace{(2(n+1))!}_{(2n+2)(2n+1)}} \frac{\cancel{(2n)!}}{\cancel{2^n} n} = \frac{2n + \cancel{2}}{(2n\cancel{+2})(2n+1)n} = \frac{1}{(2n+1)n} < 1$$

Por lo tanto, por mayorante $\sum_{n=0}^{+\infty} \partial_t (f_{2n}(x) t^n) \rightrightarrows \ell(x,t)$

5) Hacer

$u_e(x) = x$

$$\begin{cases} U \text{ de clase } C^2 \text{ en } (0,+\infty)\times(0,1) \text{ y continua en } [0,+\infty)\times[0,1] \\ U_t = U_{xx} \quad (t,x)\in(0,+\infty)\times(0,1) \\ U(x,0) = sen(\pi x) \quad x\in[0,1] \\ U(t,0) = \underbrace{U(t,L) = 0}_{-L+1} \in \mathbb{R} \quad t\in[0,+\infty) \end{cases}$$

$$U(t,x) = u_e(x) + \tilde{U}(t,x)$$

$$1 = U(t,L) = \underbrace{u_e(L)}_{L} + \tilde{U}(t,L) \quad -L+1$$