5/11

1. Hallar la solución de la ecuación de ondas,

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \quad (x,t) \in (0,L) \times (0,\infty)$$

con las condiciones de contorno:

$$\begin{cases} u(x,0) = x(L-x) & x \in [0,L] \\ u_t(x,0) = 0 & x \in [0,L] \\ u(0,t) = u(L,t) = 0 & t \in [0,\infty) \end{cases}$$

utilizando el método de separación de variables.

- $u(x,t) = X(x)\,T(t)$ $\qquad u_{tt} - c^2 u_{xx} = 0$

$$\Rightarrow \quad \frac{1}{c^2}\frac{T''(t)}{T} = \frac{X''(x)}{X} = K \quad \text{constante}$$

$$\begin{cases} T'' - c^2 K T = 0 \\ X'' - K X = 0 \end{cases}$$

$u(0,t) = 0 \to X(0) = 0$ $\quad \forall t$

$u(L,t) = 0 \to X(L) = 0$

$$\begin{cases} X'' - KX = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

- Si $K = 0$: $\quad X'' = 0 \Rightarrow X(x) = ax + b$

$0 = X(0) = b \to \boxed{b = 0}$

$0 = X(L) = La \to \boxed{a = 0}$

Solución trivial

→ Solución trivial

• $K>0$: $\lambda = \pm\sqrt{K}$

$$X(x) = A e^{\sqrt{K}x} + B e^{-\sqrt{K}x}$$

$$0 = X(0) = A + B \rightarrow B = -A$$

$$0 = X(L) = A e^{\sqrt{K}L} + B e^{-\sqrt{K}L}$$

$$0 = A\left(\underbrace{e^{\sqrt{K}L} - e^{-\sqrt{K}L}}_{\neq 0}\right) \Rightarrow \boxed{A=0} \Rightarrow \boxed{B=0}$$

La nueva:

$K = -|K|$

$X'' - KX = 0$

$K<0$: $K = -\alpha$ $\quad \alpha \in \mathbb{R}^+$

$$\lambda^2 - K = 0 \longrightarrow \lambda^2 = K = -\alpha$$

$$\lambda = \underbrace{\pm\sqrt{\alpha}\, i}_{Im\,\lambda} + \underbrace{0}_{Re\,\lambda}$$

$$X(x) = A \underbrace{e^{Re(\lambda)x}}_{1} \cos(Im(\lambda)x) + B \underbrace{e^{Re(\lambda)x}}_{1} \operatorname{sen}(Im(\lambda)\, x)$$

$$\begin{cases} X(x) = A\cos(\sqrt{\alpha}\,x) + B\operatorname{sen}(\sqrt{\alpha}\,x) \\ X(0) = 0 \\ X(L) = 0 \end{cases}$$

$0 = X(0) = A \rightarrow \boxed{A=0}$

$0 = X(L) = B \operatorname{sen}(\sqrt{\alpha}\, L)$ → $B=0$ Si $B=0$, tenemos la solución trivial.

→ $\operatorname{sen}(\sqrt{\alpha} L) = 0$

$K = -\alpha$

$\Updownarrow$

$\sqrt{\alpha}\, L = n\pi \Rightarrow \sqrt{\alpha} = \dfrac{n\pi}{L} \Rightarrow \alpha = \left(\dfrac{n\pi}{L}\right)^2$

$$\boxed{K = -\left(\frac{n\pi}{L}\right)^2}$$

$K<0$ $\begin{cases} X'' - KX = 0 \\ X(0) = 0 \\ X(L) = 0 \end{cases}$ $\rightsquigarrow$ $\boxed{X(x) = B \operatorname{sen}\left(\frac{n\pi}{L} x\right)}$

$T'' - c^2 K T = 0$

$T'' + \left(\dfrac{n\pi c}{L}\right)^2 T = 0$ ← $K = -\left(\dfrac{n\pi}{L}\right)^2$

Polinomio característico: $\lambda^2 + \left(\dfrac{n\pi c}{L}\right)^2 = 0$

$$\boxed{\lambda = \pm \frac{n\pi c}{L} i}$$

$$T(t) = C\cos\left(\frac{n\pi c}{L}t\right) + D\,\mathrm{sen}\left(\frac{n\pi c}{L}t\right)$$

$$u(x,t) = X(x)\,T(t)$$

$$= B\,\mathrm{sen}\left(\frac{n\pi}{L}x\right)\left[C\cos\left(\frac{n\pi c}{L}t\right) + D\,\mathrm{sen}\left(\frac{n\pi c}{L}t\right)\right]$$

Sean $\bar{C} = BC$, $\bar{D} = BD$.

$$u(x,t) = \mathrm{sen}\left(\frac{n\pi}{L}x\right)\left[\bar{C}\cos\left(\frac{n\pi c}{L}t\right) + \bar{D}\,\mathrm{sen}\left(\frac{n\pi c}{L}t\right)\right]$$

Teorema 0.3.

Sea $u_0 = \sum_{k=1}^{+\infty} b_k \mathrm{sen}\left(\frac{k\pi}{L}x\right)$ *y* $v_0 = \sum_{k=1}^{+\infty} b'_k \mathrm{sen}\left(\frac{k\pi}{L}x\right)$ *las condiciones iniciales del problema (0.10).*

Si

$$|b_k| < \frac{M}{k^4} \qquad |b'_k| < \frac{N}{k^3} \qquad N, M \in \mathbb{R}$$

entonces: (t, x)

$$\Rightarrow \quad U(t,x) = \sum_{k=1}^{+\infty} \mathrm{sen}\left(\frac{k\pi}{L}x\right)\left(A_k \cos\left(\frac{k\pi c}{L}t\right) + B_k \mathrm{sen}\left(\frac{k\pi c}{L}t\right)\right)$$

con $A_k = b_K$ *y* $B_k = b'_k \frac{L}{k\pi c}$ *es solución al problema (0.10).*

(x, t)

$$(0.10) \quad \begin{cases} U \text{ de clase } C^2 \text{ en } (0,+\infty)\times(0,L) \text{ y continua en } [0,+\infty)\times[0,L] \\ U_{tt} = c^2 U_{xx} \quad (t,x) \in (0,+\infty)\times(0,L) \\ U(0,x) = u_0(x) \quad x \in [0,L] \\ U_t(0,x) = v_0(x) \quad x \in [0,L] \\ U(t,0) = U(t,L) = 0 \quad t \in [0,+\infty) \end{cases}$$

$$\begin{cases} u(x,0) = x(L-x) \; [u_0(x)] & x \in [0,L] \\ u_t(x,0) = 0 \; [v_0(x)] & x \in [0,L] \\ u(0,t) = u(L,t) = 0 & t \in [0,\infty) \end{cases}$$

Si $v_0(x) = 0 \;\forall x \Rightarrow 0 = \sum_{k=1}^{+\infty} b'_k \mathrm{sen}\left(\frac{k\pi}{L}x\right)$

si y sólo si $b'_k = 0 \;\forall k$

$$u_0(x) = x(L-x) \qquad x \in (0, L)$$

$$b_K = \frac{2}{L}\int_0^L x(L-x)\, \text{sen}\left(\frac{K\pi}{L}x\right) dx$$

$$= \frac{2}{L}\left[\int_0^L Lx\, \text{sen}\left(\frac{K\pi}{L}x\right) - \int_0^L x^2\, \text{sen}\left(\frac{K\pi}{L}x\right) dx\right]$$

- $|b_K| \leq \frac{M}{K^4}$

2. *a*) Un caso particular de soluciones de la ecuación del calor son las que corresponden a situaciones en las que el perfil de temperaturas no se modifica con el tiempo (lo que equivale a decir que $u(x,t)$ no depende de t, y por lo tanto $u_t = 0$). A esa soluciones las llamaremos *soluciones estacionarias* del problema.

Hallar la solución estacionaria a la ecuación de calor, $u_e(x)$, para el problema con datos de contorno

$$u(0,t) = A, \qquad u(L,t) = B$$

Ecuación de calor: $u_t = u_{xx}$

$u_e(x)$: $\dfrac{\partial^2 u_e}{\partial x^2}(x) = 0 \ \forall x \Rightarrow u_e(x) = \alpha x + \beta$

$A = u_e(0) = \beta \rightarrow \beta = A$

$B = u_e(L) = \alpha L + A \rightarrow \alpha = \dfrac{B-A}{L}$

$$u_e(x) = \frac{B-A}{L}x + A$$

Obs: Si $B = A$, la única configuración inicial para que haya solución estacionaria es que toda la barra este a la misma temperatura *b)* Hallar la solución, $u(x,t)$, de la ecuación del calor en $(0,1) \times (0,\infty)$ con condiciones de borde

$L = 1$

$$u(0,t) = 0, \qquad u(1,t) = 1$$

L

y dato inicial

$$u_0(x) = \begin{cases} 2x & \text{si} \quad x \in [0, \frac{1}{2}] \\ 1 & \text{si} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Sugerencia: utilizar el principio de superposición de soluciones.

$u(x,t) = w(x,t) + u_e(x)$

$u_e(x) = \frac{B - A}{L} x + A \quad \to \quad u_e(x) = x$

$u(x,t) = w(x,t) + x$

$u_t - u_{xx} = 0$

$0 = u_t - u_{xx} = w_t - w_{xx} + x_t - x_{xx}$

u_e es solución de $u_t = u_{xx}$

- $w_t - w_{xx} = 0$

$$u(0,t) = 0, \qquad u(1,t) = 1$$

L

$u(x,t) = w(x,t) + x$

$u(0,t) = w(0,t) = 0$

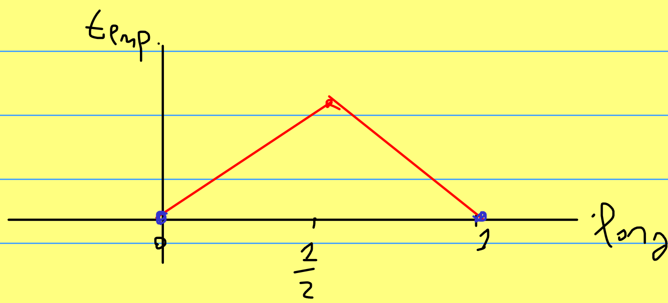
$u(1,t) = w(1,t) + 1 = 1$

$w(1,t)=0$

$$u_0(x) = \begin{cases} 2x & \text{si} \quad x \in [0, \frac{1}{2}] \\ 1 & \text{si} \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$$u_0(x) = u(0,x) = w(0,x) + x = \begin{cases} 2x & \text{si } x \in [0, 1/2] \\ 1 & \text{si } x \in [\frac{1}{2}, 1] \end{cases}$$

$$w(0,x) = \begin{cases} x & \text{si } x \in [0, 1/2] \\ 1-x & \text{si } x \in [1/2, 1] \end{cases}$$



$$\begin{cases} w_t = w_{xx} \\ w(0,t) = w(1,t) = 0 \\ u_0(x) = w(x,0) = \begin{cases} x & \text{si } x \in [0, 1/2] \\ 1-x & \text{si } x \in [1/2, 1] \end{cases} \end{cases}$$

→ Condiciones de contorno (para $w(0,t) = w(1,t) = 0$)

Condiciones inicial (para $w(x,0)$)

Teorema 0.1.

Sea $u_0(x) = \sum_{k=1}^{\infty} b_k sen\left(\frac{k\pi}{L}x\right)$ *condición inicial del problema de Cauchy-Dirichlet . Si* $\sum_{k=1}^{\infty} |b_k|$ *es convergente entonces:*

(0.8) $$U(t,x) = \sum_{k=1}^{+\infty} b_k sen\left(\frac{k\pi}{L}x\right) e^{-\left(\frac{k\pi}{L}\right)^2 t} = w(x,t)$$

es solución al problema de Cauchy-Dirichlet con condición de bordes nulas y condición inicial $u_0(x) = \sum_{k=1}^{\infty} b_k sen(kx)$*. Además* $\sum_{k=1}^{+\infty} b_k sen\left(\frac{k\pi}{L}x\right) e^{-\left(\frac{k\pi}{L}\right)^2 t}$ *converge uniformemente.*

$$a_n = 2\int_0^1 \sin(\pi x)\cos(n\pi x)\,dx$$

$$= 2\left[\frac{\sin(\pi x)\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \frac{\pi}{n\pi}\int_0^1 \cos(\pi x)\sin(n\pi x)\,dx\right]$$

$$= -\frac{2\pi}{n\pi}\left[-\cos(\pi x)\frac{\cos(n\pi x)}{n\pi}\Bigg|_0^1 - \pi\int_0^1 \sin(\pi x)\frac{\cos(n\pi x)}{n\pi}\,dx\right]$$

$$= -\frac{2}{n}\left[\left[\frac{\cos(n\pi)}{n\pi} - \frac{1}{n\pi}\right] - \frac{1}{n}\int_0^1 \sin(\pi x)\cos(n\pi x)\,dx\right]$$

$(-1)^n$

$$\int_0^1 \sin(\pi x)\cos(n\pi x)\,dx = \frac{2-2(-1)^n}{n^2\pi} + \frac{2}{n}\int_0^1 \sin(\pi x)\cos\cdots$$

$$\left(\frac{1}{n} - \frac{2}{n}\right)\int \quad = \frac{2-2(-1)^n}{n^2\pi}$$

$$a_n = \frac{2-2(-1)^n}{n^2\pi}\cdot\frac{n}{n-1}$$

$$a_n = \frac{2}{\pi}\,\frac{1-(-1)^n}{(n-1)}$$

$L = 2$

$$b_K = 2\int_0^1 w(x,0)\,\text{sen}\,K\pi x\,dx$$

$$= 2\left[\int_0^{1/2} \overset{f}{x}\,\overbrace{\text{sen}\,K\pi x}^{g'}\,dx + \int_{1/2}^{2} \overbrace{(2-x)}^{h}\,\overbrace{\text{sen}\,K\pi x}^{j'}\,dx\right]$$

$$= 2\left[-\frac{x\cos K\pi x}{K\pi}\Bigg|_0^{1/2} + \int_0^{1/2}\frac{\cos K\pi x}{K\pi}\,dx - \frac{(2-x)\cos K\pi x}{K\pi}\Bigg|_{1/2}^{2} - \int_{1/2}^{2}\frac{\cos K\pi x}{K\pi}\,dx\right]$$

$= \ldots$

$$\Rightarrow \boxed{b_K = \frac{4}{\pi^2 K^2}\,\text{sen}\left(\frac{K\pi}{2}\right)}$$

$$|b_K| = \frac{4}{\pi^2K^2}\left|\text{sen}\left(\frac{K\pi}{2}\right)\right| \leq \frac{4}{\pi^2K^2}$$

$$\sum_{K=1}^{+\infty} |b_K| \leq \frac{4}{\pi^2}\sum_{K=1}^{+\infty}\frac{1}{K^2} \quad \text{Converge}$$