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8. Estudiar la convergencia puntual y uniforme (calculando la suma) para la serie de funciones $\sum_{n=1}^{\infty} a_n(x)$ con $a_n(x) = \left(\dfrac{1-x}{1+x}\right)^n$.

x_0 fijo $\quad \displaystyle\sum_{n=1}^{\infty} \left(\frac{1-x_0}{1+x_0}\right)^n$

- $\left|\dfrac{1-x_0}{1+x_0}\right| < 1$

$|1-x_0| < |1+x_0|$ $\qquad$ • $x_0 > 0$ ✓

• $x_0 = 0$ no funciona

$|1-x_0| < |1+x_0| \leq 1 + |x_0|$

$x_0 < 0$: $1 - x_0 < 1 + |x_0|$

Cuando $x_0 < 0$: $1 - x_0 = 1 + |x_0| \Rightarrow x_0 < 0$ no funciona

Sea $r = \dfrac{1-x_0}{1+x_0}$

$\displaystyle\sum_{n=1}^{\infty} r^n \rightarrow$ Geométrica

Para saber a que converge, usar que es geométrica

$$F_n(x) = \sum_{K=1}^{n} r^K(x)$$

Queremos ver

$$\lim_n \sup_{x>0} \left| F_n(x) - F(x) \right| \overset{?}{=} 0$$

$$\lim_n \sup_{x>0} \left| \sum_{K=n+1}^{\infty} r^K(x) \right| \leq \lim_n \sup_{x>0} \sum_{K=n+1}^{\infty} \left| r(x) \right|^K$$

$$\cdot \sum_{K=0}^{+\infty} r^K(x) = \frac{1}{1-r(x)} \qquad \leq \lim_n \sup_{x>0} \frac{1}{1-r(x)}$$

$$r(x) = \frac{1-x}{1+x} > 0 \Rightarrow \sum_{K=n+1}^{+\infty} r^K(x) < \frac{1}{1-r(x)}$$

Si $x \in (0,1)$

$$\frac{1}{1-r(x)} \qquad r(x) = \frac{1-x}{1+x}$$

$$1 - r(x) = 1 - \frac{1-x}{1+x} = \frac{1+x-1+x}{1+x} = \frac{2x}{1+x}$$

$$\boxed{\frac{1}{1-r(x)} = \frac{1+x}{2x}}$$

Hay que calcular

$$\sup_{x>0} \frac{1+x}{2x}$$

Series de Fourier.

$V = \{ f : \mathbb{R} \to \mathbb{R}$ continuas a trozos y $2L$-periódicas $\}$

Espacio vectorial:

Continua salvo en finitos puntos, además donde es discontinua existen (y son finitos) los límites laterales.

$S = \left\{ \frac{1}{\sqrt{2}}, \cos x, \operatorname{sen} x, \cos 2x, \operatorname{sen} 2x, \dots \right\} \subset V$ $\quad \boxed{L = \pi}$

- $f, g \in V \Rightarrow \langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x) g(x)\, dx$

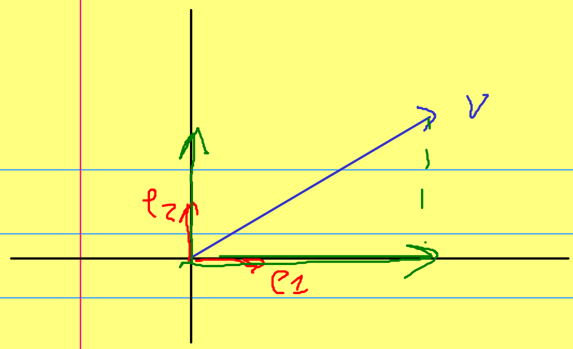
Falla que $\langle f, f \rangle = 0 \Rightarrow f = 0$

$\left(f \sim g \right.$ si $f - g = 0$ salvo en finitos puntos$\left. \right)$

Con este p.i. el conjunto S es ortonormal:

$\langle f, g \rangle = 0$ si $f \neq g$ $\quad \forall f, g \in S$

$\langle f, f \rangle = 1$



$$V = \langle V, e_1 \rangle e_1 + \langle V, e_2 \rangle e_2$$

Dada $f \in V$:

- $$S_n(x) = \frac{\overbrace{\langle f, 1 \rangle}^{a_0}}{2} + \sum_{K=1}^{n} \overbrace{\langle f, \cos Kx \rangle}^{a_K} \cos Kx + \overbrace{\langle f, \operatorname{sen} Kx \rangle}^{b_K} \operatorname{sen} Kx$$

Serie suma de Fourier.

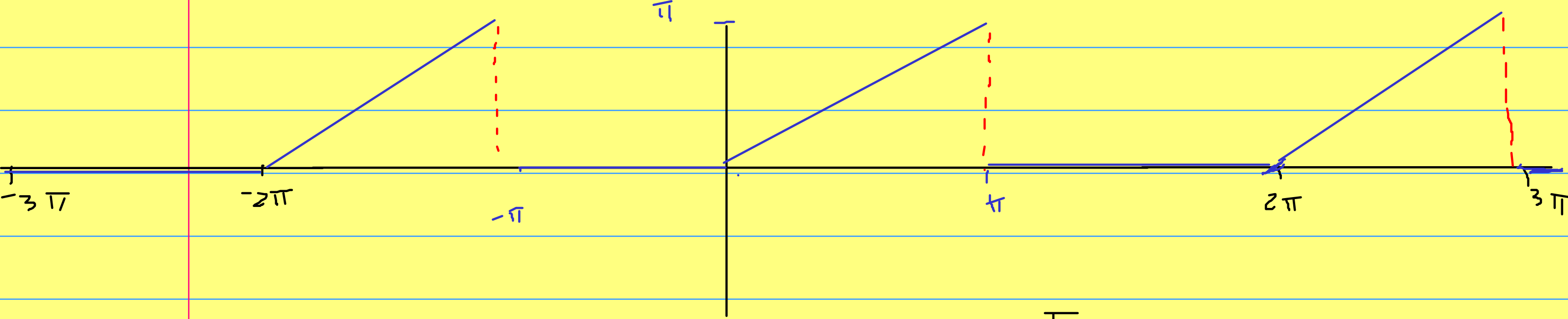
La serie de Fourier de f es:

$$S_\infty(x) = \frac{\langle f, 1 \rangle}{2} + \sum_{K=1}^{\infty} \langle f, \cos Kx \rangle \cos Kx + \langle f, \operatorname{sen} Kx \rangle \operatorname{sen} Kx$$

$$\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x) g(x)\, dx$$

$a)$ Hallar la serie de Fourier de la función 2π-peródica definida como:

$$f(x) = \begin{cases} 0 & \text{si} \quad -\pi \leqslant x < 0 \\ x & \text{si} \quad 0 \leqslant x < \pi \end{cases} .$$



$$\cdot \; a_0 = \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} x \, dx$$

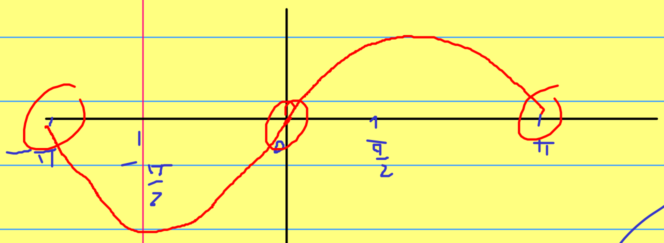
$$= \frac{1}{\cancel{\pi}} \frac{\pi^{\cancel{2}}}{2} = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$\cdot \; a_K = \langle f, \cos Kx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos Kx \, dx$$

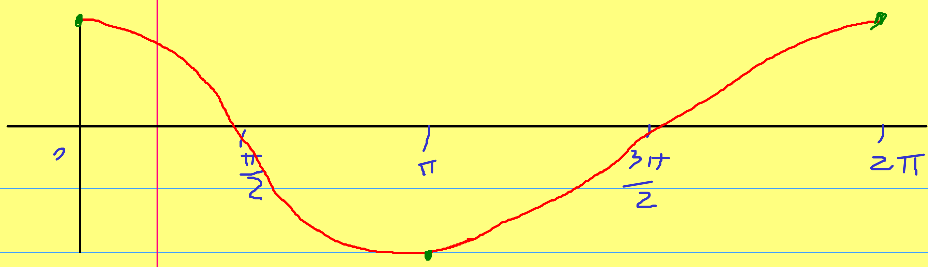
$$\int_0^{\pi} x \cos Kx \, dx = \cancel{\frac{x \operatorname{sen} Kx}{K}} \Bigg|_0^{\pi} \to 0 - \int_0^{\pi} \frac{\operatorname{sen} Kx}{K} \, dx$$

Partes



$$\int_0^{\pi} \frac{\operatorname{sen} Kx}{K} \, dx = -\frac{\cos Kx}{K^2} \Bigg|_0^{\pi} =$$

$$= -\frac{1}{K^2} \left[(-1)^K - 1 \right]$$



$\cos(K\pi) = (-1)^K$

$$\boxed{a_K = \frac{1}{\pi K^2}\left[(-1)^K - 1\right]}$$

$$f(x) = \begin{cases} 0 & \text{si } -\pi \leq x < 0 \\ x & \text{si } 0 \leq x \leq \pi \end{cases}$$

$$b_K = \langle f, \operatorname{sen} Kx \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \operatorname{sen} Kx \, dx$$

Por cómo es $f(x)$

$$= \frac{1}{\pi}\int_0^{\pi} x \operatorname{sen} Kx \, dx$$

$$\int_0^{\pi} \underbrace{x}_{f} \, \underbrace{\operatorname{sen} Kx}_{g'} \, dx = -\frac{x\cos Kx}{K}\Bigg|_0^{\pi} + \int_0^{\pi} \frac{\cos Kx}{K} \, dx$$

$$= -\frac{1}{K}\left[\pi \underbrace{\cos K\pi}_{(-1)^K} - 0\right] + \int_0^{\pi} \frac{\cos Kx}{K} \, dx$$

$$= -\frac{\pi}{K}(-1)^K + \frac{\operatorname{sen} Kx}{K^2}\Bigg|_0^{\pi} \rightarrow 0$$

$$\Rightarrow b_K = \frac{1}{\pi}\left(-\frac{\pi}{K}(-1)^K\right) = -\frac{1}{K}(-1)^K = \frac{(-1)^{K+1}}{K}$$

$$\boxed{b_K = \frac{(-1)^{K+1}}{K}}$$

$$S(x) = \overset{\frac{a_0}{2}}{\frac{\pi}{4}} + \sum_{K=1}^{\infty} \overset{a_K}{\frac{(-1)^K - 1}{\pi K^2}} \cos Kx + \overset{b_K}{\frac{(-1)^{K+2}}{K}} \operatorname{sen} Kx$$

b) Sustituyendo x por π, demostrar que:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$S(\pi) = \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{(-1)^K - 1}{K\pi^2} (-1)^K$$

$$= \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{1 - (-1)^K}{K\pi}$$

Teo (Dini): $f: \mathbb{R} \to \mathbb{R}$ continua a trozos, 2π periódica tal que $\forall x \in \mathbb{R}$ los límites

$$\lim_{t \to 0^+} \frac{f(x+t) - f(x^+)}{t}, \quad \lim_{t \to 0^-} \frac{f(x+t) - f(x^-)}{t}$$

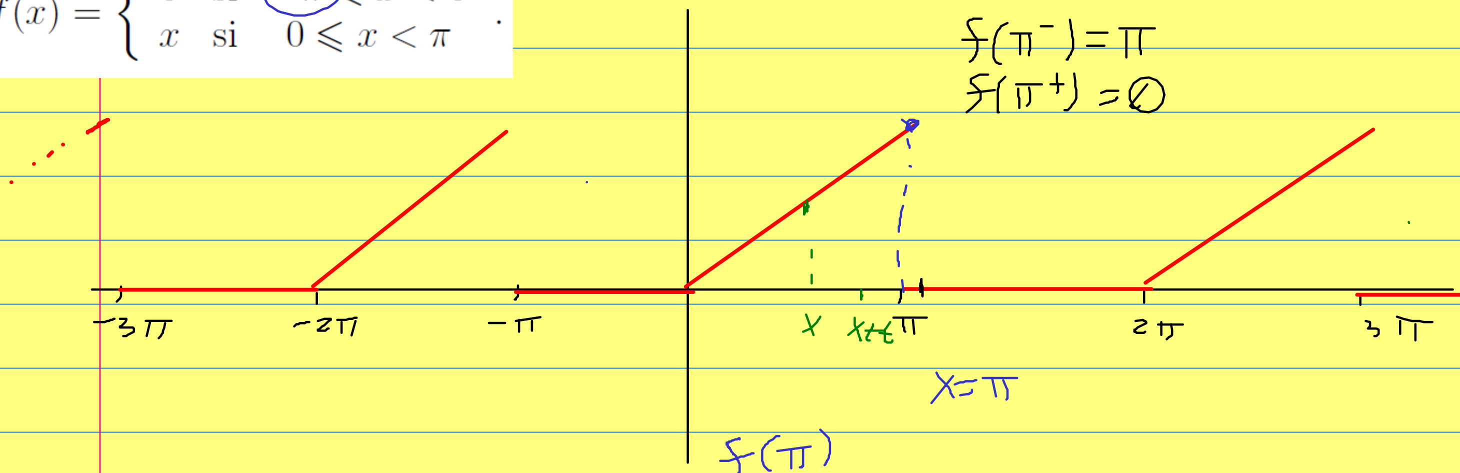
existen.

Entonces $\lim_n S_n(x) = \dfrac{f(x^-) + f(x^+)}{2}$

$$\left(S_n(x) \xrightarrow{c.p.} \frac{f(x^-) + f(x^+)}{2} \right.$$

Veamos que $\forall x \in \mathbb{R}$: $\displaystyle\lim_{t \to 0^+} \frac{f(x+t) - f(x^+)}{t}$ existe

$$f(x) = \left\{ \begin{array}{lll} 0 & \text{si} & -\pi \leqslant x < 0 \\ x & \text{si} & 0 \leqslant x < \pi \end{array} \right. .$$



- Si x está en un segmento horizontal: $f(x+t) - f(x^+) = 0$

  por lo tanto $\displaystyle\lim_{t \to 0^+} \frac{f(x+t) - f(x^+)}{t} = 0$

- Si x está en un segmento no horizontal ($x \neq \pi$)

$$\left.\begin{array}{l} f(x+t) = x+t \\ f(x^+) = x^+ \end{array}\right\} f(x+t) - f(x^+) = \cancel{x} + t - \cancel{x}^+ = t$$

$$\lim_{t \to 0^+} \frac{f(x+t) - f(x^+)}{t} = \lim_{t \to 0^+} \frac{t}{t} = 1$$

- Entonces por Dini:

$$S(x) = \frac{f(x^-) + f(x^+)}{2}$$

$$S(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{\pi - 0}{2} = \frac{\pi}{2}$$

$$S(\pi) = \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{(-1)^K - 1}{K\pi^2} (-1)^K$$

$$= \frac{\pi}{4} + \sum_{K=1}^{\infty} \frac{1 - (-1)^K}{K^2 \pi}$$

$$\frac{\pi}{2} = S(\pi) = \frac{\pi}{4} + \frac{1}{\pi} \cdot \sum_{K=1}^{\infty} \frac{1 - (-1)^K}{K^2}$$

Serie abs. convergente

$$\frac{\pi^2}{4} = \sum_{K=1}^{\infty} \frac{1 - (-1)^K}{K^2} = \sum_{K\,par} \frac{1 - (-1)^K}{K^2} = 0 \text{ si } K \text{ es par.}$$

$$+ \sum_{K\,impar} \frac{1 - \overset{-1}{(-1)^K}}{K^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$\frac{\pi^2}{4} = \sum_{K\,impar} \frac{2}{K^2} = 2 \sum_{K\,impar} \frac{1}{K^2}$$

$$= 2 \sum_{K=1}^{\infty} \frac{1}{(2K-1)^2}$$

$$\boxed{\frac{\pi^2}{8} = \sum_{K=1}^{\infty} \frac{1}{(2K-1)^2}}$$

c) A partir de lo anterior, concluir que:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ es absolutamente convergente.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ par}} \frac{1}{n^2} + \sum_{n \text{ impar}} \frac{1}{n^2}$$