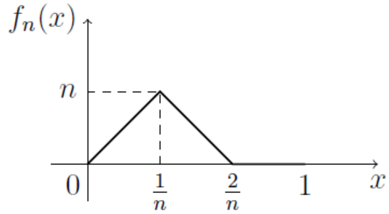
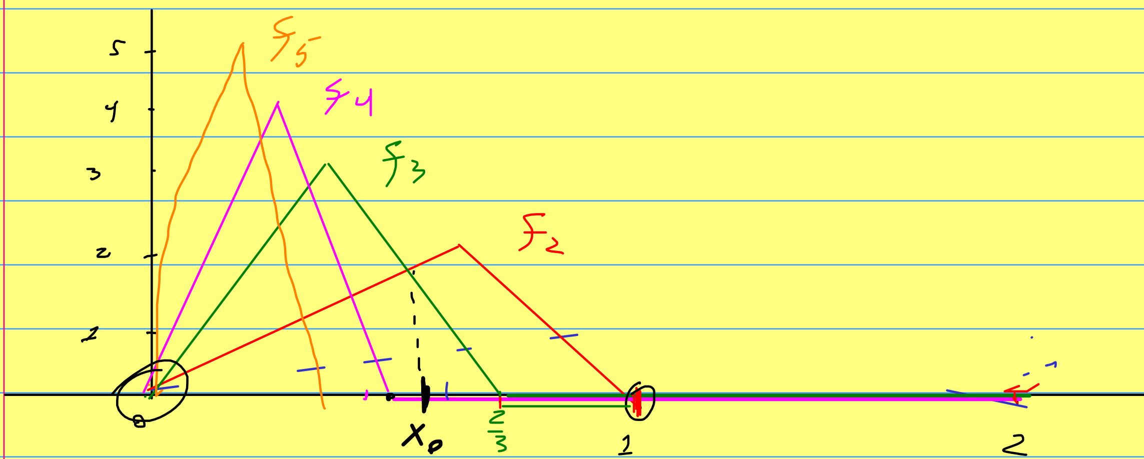
27/10

6. Sea $f_n : [0, \cancel{3}\ 2] \to \mathbb{R}$ definida como en la figura:

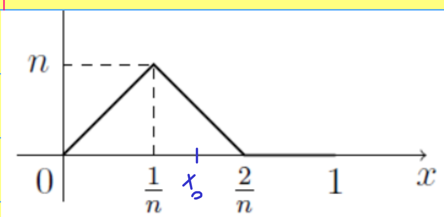




a) Demostrar que $f_n \to 0$ (puntualmente).

$n \geq 2$

$x_0 \in [0, 1]$ fijo : $\lim_n f_n(x_0) = 0$

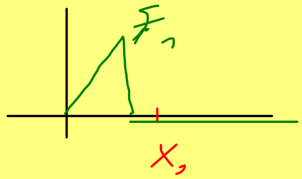


Si $x_0 \in \{0, 1\} \Rightarrow f_n(x_0) = 0 \ \forall n \geq 2 \Rightarrow \lim_n f_n(x_0) = 0$

Si $x_0 \in (0, 1)$: $\lim_n f_n(x_0) = 0$ quiere decir:

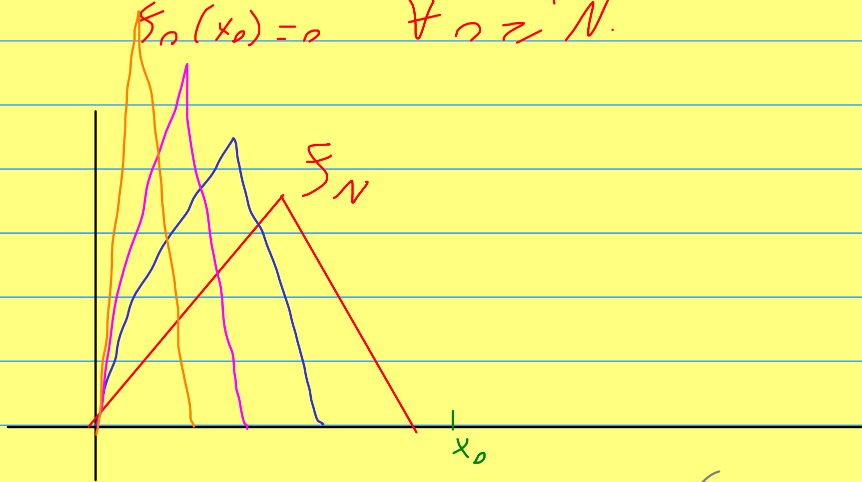
$\forall \varepsilon > 0$, $\exists n_0(\varepsilon)$ tal que si $n \geq n_0$

entonces $|f_n(x_0) - 0| < \varepsilon$



Obs: i) $f_n(x_0)=0$ si y sólo si x_0 no está en la base del triángulo de f_n.

ii) Si $f_N(x_0)=0$ para algún N, entonces $f_n(x_0)=0 \quad \forall n \geq N$.



$x_0 \in (0,2)$

Para probar que $\lim_n f_n(x_0)=0$, nos alcanza $\boxed{\text{encontrar } n_0 \in \mathbb{N} \text{ tal que } f_{n_0}(x_0)=0}$

De ser así: $\underbrace{\forall n \geq n_0 \qquad f_n(x_0)=0}_{\text{Obs } 2}$

$f_2(x_0), f_3(x_0), \ldots\ldots, f_{n_0}(x_0)=0, 0, 0, 0, 0, \ldots$

Entonces $\lim_n f_n(x_0)=0$

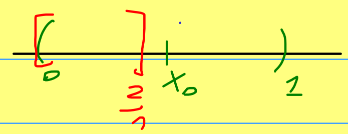
- $x_0 \in (0,1)$, $x_0 > 0$

- La base del triángulo f_n es $[0, \frac{2}{n}]$

obs 1

$f_n(x_0) = 0 \iff x_0$ no está en la base de f_n $\iff x_0 \notin [0, \frac{2}{n}]$

$\left.\begin{array}{l} x_0 \in (0,1) \\ x_0 \notin [0, \frac{2}{n}] \end{array}\right\}$

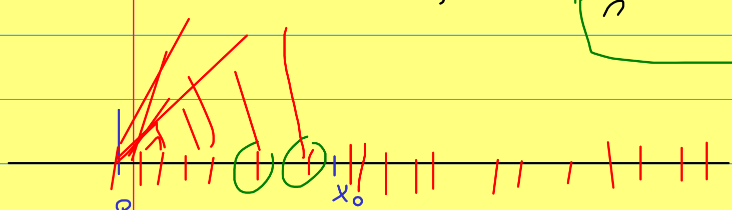


$x_0 > \frac{2}{n}$

Tenemos que encontrar $n \in \mathbb{N}$ tal que $\boxed{x_0 > \frac{2}{n}}$

Con x_0 fijo, ¿Cómo encontramos n tal que $x_0 > \frac{2}{n}$?

- Recordar que $\boxed{\frac{2}{n} \xrightarrow[n]{} 0}$, $\forall \varepsilon > 0$, $\exists n_0(\varepsilon)$ tq $\frac{2}{n} < \varepsilon \quad \forall n \geq n_0$



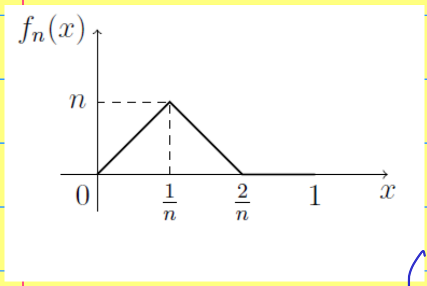
Tomo $\varepsilon = x_0$.

Ya está.

Obs: Podemos tomar $\boxed{n = \left\lceil \frac{2}{x_0} \right\rceil + 1}$

$b)$ Probar que $\overset{1}{\lim \int_0^1 f_n} \neq \int_0^1 \underbrace{\lim f_n}_{} = 0$.

$$\int_0^1 \lim_n f_n(x)\,dx = \int_0^1 0\,dx = 0$$



$$\int_0^1 f_n(x)\,dx = \frac{\overset{\text{base}}{\frac{2}{n}} \cdot \overset{\text{altura}}{n}}{2} = \frac{2n}{2n} = 1 \quad \forall n$$

$$\lim_n \int_0^1 f_n(x)\,dx = \lim_n 1 = 1$$

Prop: Si $f_n \rightrightarrows f$ entonces $\lim_n \int_a^b f_n(x)\,dx = \int_a^b \overset{f}{\lim_n f_n(x)}\,dx$

$c)$ Calcular el supremo de $f_n(x)$ para $x \in [0,1]$.

- Candidatos a máximo:

$f_n'(x) = 0$, $\quad \nexists\, f_n'(x)$, $\quad x = 0$, $\quad x = 1$

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7. Estudiar la convergencia puntual y uniforme de las series de funciones $\sum_{n=1}^{\infty} a_n(x)$ cuando:

a) $a_n(x) = \frac{\operatorname{sen}(nx)}{n^2}$

$$F_n(x) = \sum_{K=1}^{n} a_n(x)$$

$F_n : U \subset \mathbb{R} \to \mathbb{R}$

- $F_n \xrightarrow{c.p} F$: Dado $x_0 \in U$ fijo, $\lim_{n} |F_n(x_0) - F(x_0)| = 0$

- $F_n \xrightarrow{c.u.} F$: Si $\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon)$ tal que

$$|F_n(x) - F(x)| < \varepsilon \quad \forall n \geq n_0(\varepsilon) \quad \forall x \in U$$

- $\lim_{n} \sup_{x \in U} |F_n(x) - F(x)|$

- Criterio mayorante de Weierstrass
  M-test

  Si $\exists \{A_n\} \subset \mathbb{R}_{\geq 0}$ tal que $|F_n(x)| \leq A_n$

  y además $\sum_{n=1}^{\infty} A_n$ → Serie converge entonces

  F_n converge uniformemente

a) $f_n(x) = \dfrac{\text{sen}(nx)}{n^2}$, $F_n(x) = \sum_{K=1}^{n} \dfrac{\text{sen}(nK)}{K^2}$

$$|f_n(x)| = \left|\frac{\text{sen}(nx)}{n^2}\right| = \frac{|\text{sen}(nx)|}{|n^2|} \leq \frac{1}{n^2} =: A_n$$

- $\displaystyle\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converge

Por el M-test $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converge uniformemente.

(en particular converge puntualmente)

c) $a_n(x) = \dfrac{x^n}{n!}$

$x_0 \in \mathbb{R}$ fijo: $\displaystyle F_n(x_0) = \sum_{k=1}^{n} \frac{x_0^k}{k!}$, $\dfrac{|x_0|^k}{k!}$

Queremos ver si $\displaystyle\sum_{k=1}^{\infty} \frac{x_0^k}{k!}$ converge ($x_0 \in \mathbb{R}$)

$$\frac{\cancel{k!}}{(k+1)\cancel{k!}}$$

$$\frac{\dfrac{x_0^{k+1}}{(k+1)!}}{\dfrac{x_0^k}{k!}} = \frac{x_0^{\cancel{k}+1}}{\cancel{x_0^k}} \frac{k!}{(k+1)!} = \boxed{\frac{x_0}{k+1}}$$

- $\dfrac{x_0}{k+1} < 1$ a partir de algún k

$x_0 > 0$

Para $x_0 \in \mathbb{R}$: considere $\displaystyle\sum_{k=1}^{n} \boxed{\frac{|x_0|^k}{k!}}$ a_n

$$\frac{a_{k+1}}{a_k} = \cdots = \frac{|x_0|}{k+1} < 1 \quad \text{a partir de um } k$$

Obs:

$$0 < |F_n(x)| = \left| \sum_{k=1}^{n} \frac{x_0^k}{k!} \right| \leq \sum_{k=1}^{n} \frac{|x_0|^k}{k!}$$

$F_n \xrightarrow{c.u.} F$ sii

$$\lim_n \sup_{x \in \mathbb{R}} \left| F_n(x) - F(x) \right| = 0$$

$$|F_n(x) - F(x)| = \left| \sum_{k=1}^{n} \frac{x^k}{k!} - \sum_{k=1}^{\infty} \frac{x^k}{k!} \right|$$

$$= \left| - \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right|$$

$$= \sum_{k=n+1}^{\infty} \frac{x^k}{k!}$$

$$\lim_n \Big(\underbrace{\sup_{x \in \mathbb{R}} \sum_{k=n+1}^{\infty} \frac{x^k}{k!}}_{\text{dá "infinito"}} \Big) \text{ "=" } \infty, \quad F_n \not\xrightarrow{c.u.} F$$

$$F_n(x) = \sum_{k=1}^{n} \frac{\operatorname{sen}(nx)}{n^2}$$

$$|F_n - F| = \sum_{k=n+1}^{\infty} \frac{\operatorname{sen}(kx)}{k^2}$$

$$\sup \sum_{k=n+1}^{\infty} \frac{\operatorname{sen}(kx)}{k^2} = \sum_{k=n+1}^{\infty} \frac{1}{k^2}$$

$$\lim_{n} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0$$