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$\dot{x} = Ax$

Complejos

$\dot{y} = By$

$r_0 = \sqrt{u_0^2 + v_0^2}$

$\theta_0 = \arctan\left(\frac{u_0}{v_0}\right)$

$x(t) = P\,y(t)$

$y(t) =$

$y(t) = (r_0 e^{at}\cos(-bt+\theta_0), r_0 e^{at}\sin(-bt+\theta_0))$

$$x(t) = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{P} y(t) = r_0 e^{at}\begin{pmatrix} a\cos(-bt+\theta_0) + b\,\mathrm{sen}(-bt+\theta_0) \\ c\cos(-bt+\theta_0) + d\,\mathrm{sen}(-bt+\theta_0) \end{pmatrix}$$

$$\underset{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}{\overset{\parallel}{x(0)}} = r_0 \begin{pmatrix} a\cos(\theta_0) + b\,\mathrm{sen}(\theta_0) \\ c\cos(\theta_0) + d\,\mathrm{sen}(\theta_0) \end{pmatrix}$$

12. (a) Dada una matriz A, probar que si α no es valor propio entonces la ecuación $x' = Ax + e^{\alpha t}b$ tiene una única solución de la forma $x(t) = e^{\alpha t}u$ con $u \in \mathbb{R}^n$. Calcular u en función de A, b y α.

(b) Resolver el sistema:

$$\begin{aligned} x' &= \overbrace{2x + y}^{Ax} + e^{2t} \\ y' &= x + 2y - e^{2t} \end{aligned}$$

$b_0 = 1$

$b_1 = -1$

$A \in M_{n\times n}$

$b \in \mathbb{R}^n$

$$x' = Ax + e^{\alpha t} b \quad , \quad b = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$$

a) i) $x(t) = e^{\alpha t}u$ es solución, $u, v \in \mathbb{R}^n$

ii) Supongamos $x_1(t) = e^{\alpha t}u$ y $x_2(t) = e^{\alpha t}v$

son soluciones de $\boxed{\dot{x} = Ax + e^{\alpha t} b}$

$\dot{x}_1 = \alpha e^{\alpha t} u \rightarrow \alpha e^{\alpha t} u = A e^{\alpha t} u + e^{\alpha t} b$

$\dot{x}_2 = \alpha e^{\alpha t} v \rightarrow \alpha e^{\alpha t} v = A e^{\alpha t} v + e^{\alpha t} b$

- Def: Si $D \in \mathcal{M}_{n\times n}(\mathbb{R})$ entonces definimos

$$\mathcal{M}_{n\times n}(\mathbb{R}) \ni e^{D} := \sum_{K=0}^{+\infty} \frac{D^K}{K!}$$ entendiendo $D^0 = I_{n\times n}$

Taylor de e^x alrededor de 0 : $\sum_{K=0}^{+\infty} \frac{x^K}{K!}$

- Diagonalizable :

$$D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

$$D^K = \operatorname{diag}(\lambda_1^K, \ldots, \lambda_n^K)$$

Ej: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$

Si $D = diag(\lambda_1, \ldots, \lambda_n)$, entonces

$$A = e^D := \sum_{k=0}^{+\infty} \frac{D^k}{k!} = \sum_{k=0}^{+\infty} \frac{diag(\lambda_1^k, \ldots, \lambda_n^k)}{k!}$$

$a_{ij} = 0$ si $i \neq j$

$$a_{ii} = \sum_{k=0}^{+\infty} \frac{\lambda_i^k}{k!} = e^{\lambda_i}$$

Taylor de e^{λ_i} alrededor de 0.

Si $D = diag(\lambda_1, \ldots, \lambda_n)$, entonces

$$e^D = diag(e^{\lambda_1}, \ldots, e^{\lambda_n})$$

Ej: $D = diag(-1, 0, 3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$e^D = diag(e^{-1}, \underset{1}{\underset{\parallel}{e^0}}, e^3) = \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^3 \end{pmatrix}$$

• Si $A = P^{-1} B P$, ¿Cómo se vinculan e^A con e^B? — B diagonal

Obs: Si $A = P B P^{-1} \to A^K = P B^K P^{-1}$

Demo: $K=1$ ✓

Paso inductivo: Suponemos $A^n = P B^n P^{-1}$, queremos probar que $A^{n+1} = P B^{n+1} P^{-1}$

$$A^{n+1} = A^n A = (P B^n P^{-1})(P\ B\ P)$$
$$= P B^n (\underbrace{P^{-1} P}_{I}) B P$$
$$= P \underbrace{B^n B}_{B^{n+1}} P^{-1} = P B^{n+1} P^{-1}$$

$A = P B P^{-1}$

$$e^A = \sum_{K=0}^{+\infty} \frac{A^K}{K!} = \sum_{K=0}^{+\infty} \frac{(P B P^{-1})^K}{K!} \overset{\text{obs}}{=} \sum_{K=0}^{+\infty} \frac{P B^K P^{-1}}{K!}$$

P, P^{-1} no dependen de K y están multiplicando todos los sumandos

$$= P \underbrace{\left(\sum_{K=0}^{+\infty} \frac{B^K}{K!} \right)}_{e^B} P^{-1} \to \boxed{e^A = P e^B P^{-1}}$$

$t \in \mathbb{R}$

Obs: $D = diag(\lambda_1, \ldots, \lambda_n) \Rightarrow Dt = diag(\lambda_1 t, \ldots, \lambda_n t)$

$\Rightarrow e^{Dt} = diag(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$

9. 1) $A = \begin{pmatrix} 4 & -6 \\ 2 & -4 \end{pmatrix}$

Hasta ahora solo sabemos calcular e^{Dt} cuando D es diagonal, pero A NO es diagonal, entonces vemos si es diagonalizable.

- $p(\lambda) = (4-\lambda)(-4-\lambda) + 12$
$= \lambda^2 - 16 + 12$
$= \lambda^2 - 4$

Los valores propios son 2 y -2

$A = P D P^{-1}$ con $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

- P:

$v_1 \in Ker(A - 2I)$

1) $A = \begin{pmatrix} 4 & -6 \\ 2 & -4 \end{pmatrix}$ | $A - 2I = \begin{pmatrix} 2 & -6 \\ 2 & -6 \end{pmatrix}$

$2x - 6y = 0 \rightsquigarrow \boxed{x = 3y}$ $\boxed{v_1 = (3, 1)}$

$v_2 \in Ker(A + 2I)$ | $A + 2I = \begin{pmatrix} 6 & -6 \\ 2 & -2 \end{pmatrix}$

$6x - 6y = 0 \to \boxed{x = 1} \to \boxed{v_2 = (1,1)}$

$$P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

- $e^{At} = P \, e^{Dt} P^{-1}$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \to e^{Dt} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$$

Si $P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow P^{-1} = \overset{\frac{1}{2}}{\frac{1}{\det(P)}} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$$

Para terminar: Hacer el producto.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det A \neq 0$

$\det A = ad - bc$

Recordar: Al escalerizar se puede:

- Intercambiar dos filas o columnas
- Sumar " " " "
- Multiplicar una " " " por un escalar

6) $A = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}$

$$e^A = \sum_{K=0}^{+\infty} \frac{A^K}{K!}.$$

- $B \cdot C = C B \to e^{B+C} = e^B \cdot e^C$

$$A = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{C} + \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}}_{B}$$

Verificar $CB = BC$:

$$e^A = e^{B+C} = e^{B} e^{C}$$

$B \to$ Diagonal $\to e^B = \begin{pmatrix} e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^\lambda \end{pmatrix}$

$$e^C := \sum_{K=0}^{+\infty} \frac{C^K}{K!}$$

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to C^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^C = \sum_{K=0}^{2} \frac{C^K}{K!} + \underbrace{\boxed{\sum_{K=3}^{+\infty} \frac{C^K}{K!}}}_{=0} \quad C^K = 0$$

$$e^C = Id + \frac{C}{1!} + \frac{C^2}{2!} =$$

Nilpotente

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \to e^A = \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix}}^{e^C} \overbrace{\begin{pmatrix} e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^\lambda \end{pmatrix}}^{e^D}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix} e^\lambda Id$$

$$= \begin{pmatrix} e^\lambda & 0 & 0 \\ e^\lambda & e^\lambda & 0 \\ \frac{e^\lambda}{2} & e^\lambda & e^\lambda \end{pmatrix}$$

$$At = \begin{pmatrix} \lambda t & 0 & 0 \\ t & \lambda t & 0 \\ 0 & t & t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ 0 & t & 0 \end{pmatrix}}_{C} + \underbrace{\begin{pmatrix} \lambda t & 0 & 0 \\ 0 & \lambda t & 0 \\ 0 & 0 & \lambda t \end{pmatrix}}_{B}$$

$$e^B = e^{\lambda t} I$$

$C^0 = Id$

$$C^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t^2 & 0 & 0 \end{pmatrix}, \quad C^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \forall k \geq 3$$

$$e^C = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{t^2}{2} & t & 1 \end{pmatrix} = \overset{C^0}{Id} + C + \frac{C^2}{2!}$$

$$e^A = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{pmatrix} e^{\lambda t}$$

$$J = \begin{pmatrix} \lambda & 0 & 0 & \cdots & & 0 \\ 1 & \lambda & 0 & \cdot & \cdot & \vdots \\ 0 & 1 & \lambda & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & \cdot & & 1 & \lambda \end{pmatrix} \quad n \times n$$

$$e^{J} = e^{\lambda t} \begin{pmatrix} 1 & & & & 0 \\ t & 1 & & & \\ \frac{t^2}{2} & t & 1 & & \\ \vdots & & & 1 & \\ \vdots & & & \cdot & 1 \\ \frac{t^{n-1}}{(n-1)!} & \cdots & \frac{t^2}{2} & t & 1 \end{pmatrix}$$