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5. Sea A una matriz 2×2, con valores propios reales λ y μ diferentes. Supongamos que $(0,1)$ y $(-1,1)$ son vectores propios asociados a los valores propios λ y μ respectivamente. Esquematizar el diagrama de fase para los siguientes casos:

(a) $0<\lambda<\mu$ (b) $0<\mu<\lambda$ (c) $\lambda<\mu<0$

(d) $\lambda<0<\mu$ (e) $\mu<0<\lambda$ (f) $0=\lambda, \mu>0$

① $\dot{x} = Ax$

Problema original

$\dot{y} = By$ ② $\binom{u}{v}$

Hallamos las soluciones

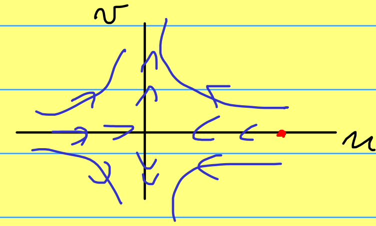
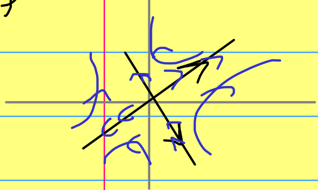


Diagrama que queremos

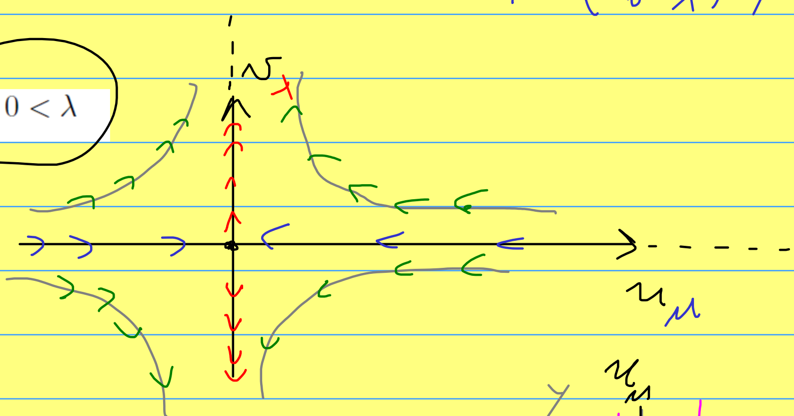


$x(t) = P\,y(t)$ ③

$\dot{y} = \begin{pmatrix}\mu & 0\\ 0 & \lambda\end{pmatrix} y \to u(t) = u_0 e^{\mu t}$

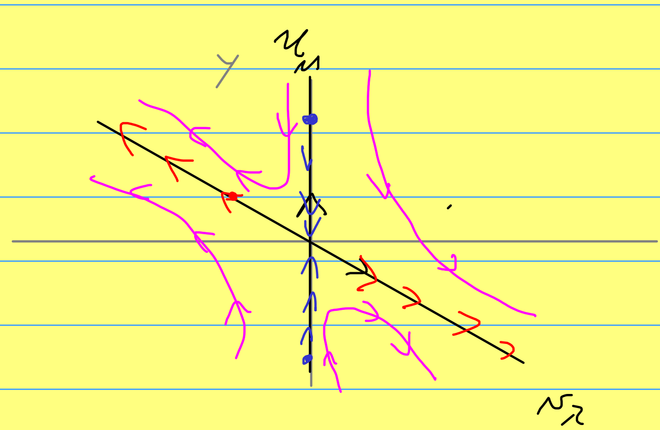
$v(t) = v_0 e^{\lambda t}$

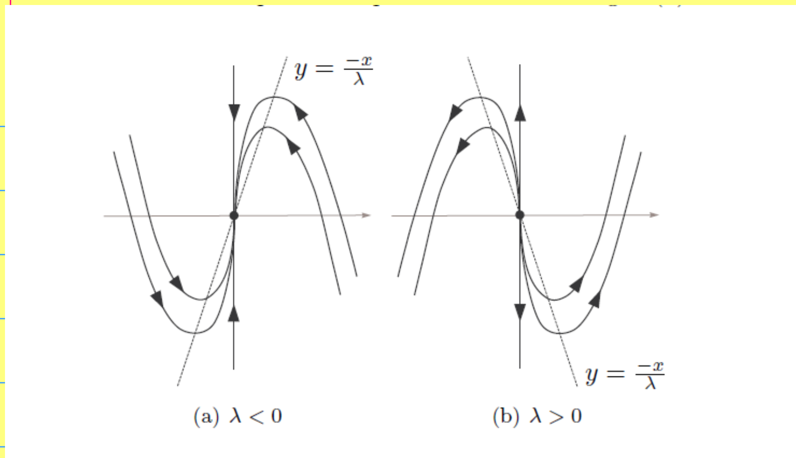
(e) $\mu<0<\lambda$



$u_\mu = (0,1)$

$v_\lambda = (-1,1)$



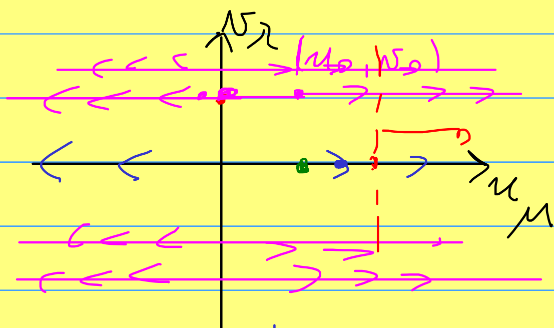


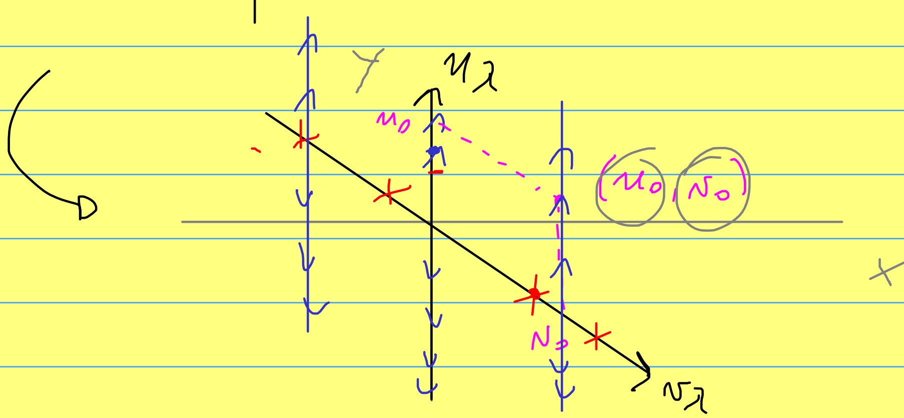
(a) $\lambda < 0$ (b) $\lambda > 0$

Jordan

(f) $0 = \lambda, \mu > 0$

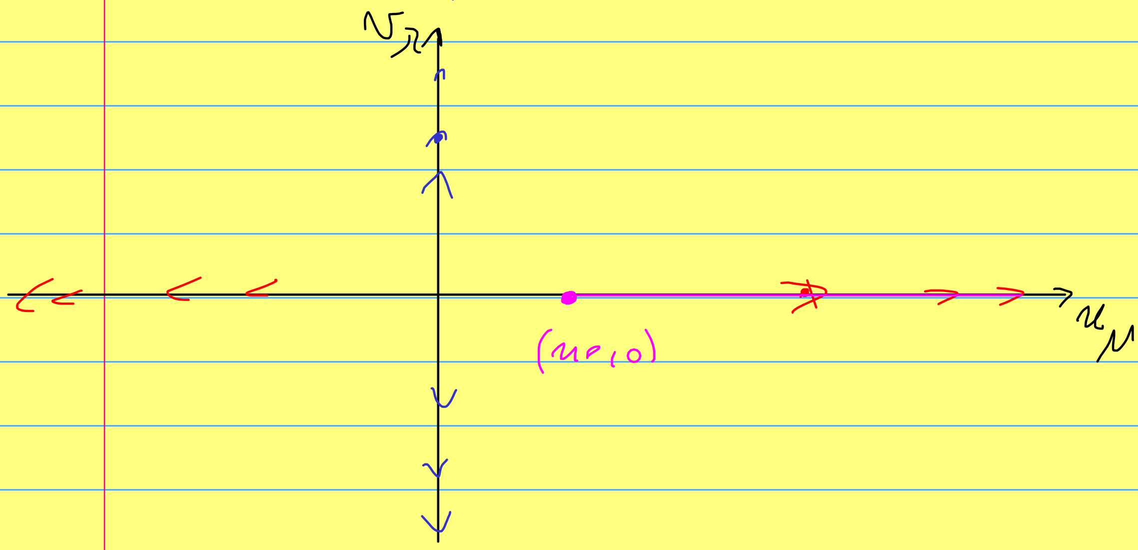
$$\begin{cases} u' = \mu u \to u(t) = u_0 e^{\mu t} \\ v' = 0 \to v(t) = v_0 \end{cases}$$





(a) $0 < \lambda < \mu$

$$\dot{y} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} y \to \begin{cases} u(t) = u_0 e^{\mu t} \\ v(t) = v_0 e^{\lambda t} \end{cases}$$



$u(t) = u_0 e^{\mu t} \to \frac{u}{u_0} = e^{\mu t} \to \mu t = \ln\left(\frac{u}{u_0}\right)$

$v(t) = v_0 e^{\lambda t}$

$$\boxed{\to t = \frac{1}{\mu} \ln\left(\frac{u}{u_0}\right)}$$

$v = v_0 \, e^{\overset{a}{\frac{\lambda}{\mu}} \ln\left(\overset{b}{\frac{u}{u_0}}\right)}$

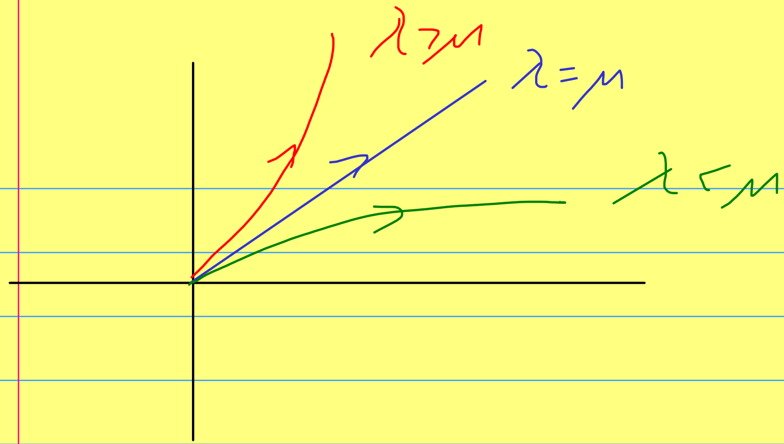
$e^{a \ln(b)} = b^a$

$v = \left(\frac{v_0}{u_0^{\lambda/\mu}}\right) u^{\frac{\lambda}{\mu}}$

$\to \lambda = \mu \to \frac{\lambda}{\mu} = 1 \to v = m u$ recta

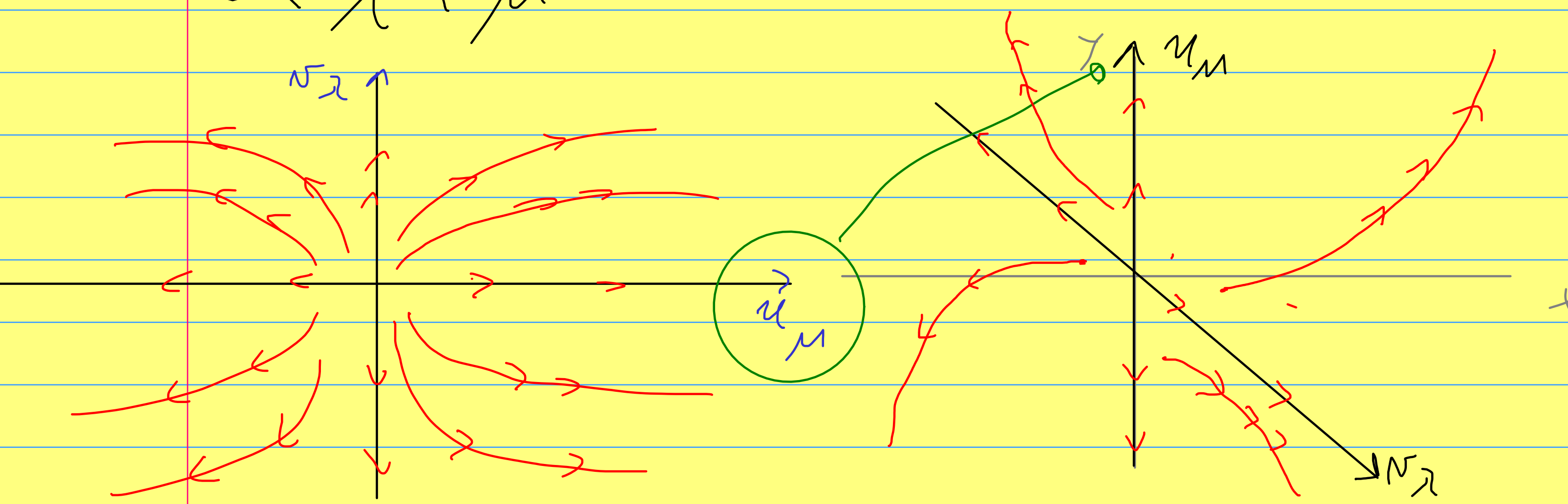
$\to \lambda > \mu \Rightarrow \frac{\lambda}{\mu} > 1 \to v = K u^{\frac{\lambda}{\mu}}$ "parábola"

$\to \lambda < \mu \Rightarrow \frac{\lambda}{\mu} < 1 \to v = K u^{\frac{\lambda}{\mu}}$ "raíz"



$\lambda, \mu > 0$
$\lambda, \mu < 0$

$0 < \lambda < \mu$



8. Escribir las siguientes ecuaciones diferenciales lineales a coeficientes constantes en la forma $x' = Ax$, y resolver:

(a) $x'' + x' - 2x = 0$ (b) $x'' + x = 0$ (c) $x''' - 2x'' - x' + 2x = 0$

$x' = y$

Definimos: $y = x'$ $\rightarrow$ $y' = x'' = -x' + 2x$
$= -y + 2x$

$z = x''$

$x' = y$

$x'' = z = y'$

$z' = x''' = 2x'' + x' - 2x = 2z + y - 2x$

$$\begin{cases} x' = y \\ y' = z \\ z' = 2z + y - 2x \end{cases}$$

$z = x''$ $\qquad z' = x'''$

$x(t)$
$y(t)$
$z(t)$

Sea $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ $\rightsquigarrow$ $A \in M_{3\times 3}(\mathbb{R})$ t.q. $X' = AX$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

1°) $p(\lambda) = \det \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & 1 & 2-\lambda \end{pmatrix}$

$= -\lambda\left[-\lambda[2-\lambda] - 1\right] - 1(2) + 0(-)$

$= \lambda^2(2-\lambda) + \lambda - 2$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$-\cancel{1} + \cancel{2} + \cancel{1} \cancel{-2}$$

$\lambda = 1$ es raíz evidente

$$\begin{array}{c|c|c|c|c}
 & -1 & 2 & 1 & -2 \\
1 & & -1 & 1 & 2 \\
\hline
 & -1 & 1 & 2 & 0
\end{array}$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = (\lambda - 1)(-\lambda^2 + \lambda + 2)$$

$$\begin{array}{c|c|c|c}
 & -1 & 1 & 2 \\
-2 & & 1 & -2 \\
\hline
 & -1 & 2 & 0
\end{array}
\quad \to -\lambda^2 + \lambda + 2 = (\lambda + 1)\overbrace{(-\lambda + 2)}^{\lambda - 2}$$

Los valores propios son: -1, 1, $+2$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

- $$\begin{cases} u' = u \\ v' = -v \\ w' = 2w \end{cases} \to Y(t) = \begin{pmatrix} u_0 e^{t} \\ v_0 e^{-t} \\ w_0 e^{2t} \end{pmatrix}$$

- **Vectores propios:**

$\underline{\lambda = 1}$: $v_1 \in \mathrm{Ker}(A - I)$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \to A - I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$\begin{cases} -x + y = 0 \rightarrow x = y \\ -y + z = 0 \rightarrow y = z \\ -2x + \overbrace{y + z}^{2x} = 0 \end{cases}$$

$$Ker(A - I) = \{(x, x, x) : x \in \mathbb{R}\} \rightarrow \boxed{V_2 = (2, 2, 2)}$$

$\lambda = -1 \quad V_2 \in Ker(A + I)$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \rightarrow A + I = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & 1 & 3 \end{pmatrix}$$

$$\begin{cases} x + y = 0 \rightarrow x = -y \\ y + z = 0 \rightarrow y = -z \\ \underbrace{\underbrace{-2x}_{} + y}_{3y} + \underbrace{3z}_{-3y} = 0 \end{cases} \Bigg\} \boxed{x = z}$$

$$Ker(A + I) = \{(x, -x, x) : x \in \mathbb{R}\}$$

$$\boxed{V_2 = (1, -1, 1)}$$

$\underline{\lambda=2}$: $v_3 \in \operatorname{Ker}(A-2I)$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \rightarrow A-2I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} -2x + y = 0 & \rightarrow y = 2x \\ -2y + z = 0 & \rightarrow z = 2y = 4x \end{cases}$$

$$\operatorname{Ker}(A-2I) = \{(x, 2x, 4x) : x \in \mathbb{R}\}$$

$$\boxed{v_3 = (1, 2, 4)}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}$$

$$x(t) = P\,y(t) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} u_0 e^{t} \\ v_0 e^{-t} \\ w_0 e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} u_0 e^{t} + v_0 e^{-t} + w_0 e^{2t} \\ u_0 e^{t} - v_0 e^{-t} + 2 w_0 e^{2t} \\ u_0 e^{t} + v_0 e^{t} + 4 w_0 e^{2t} \end{pmatrix} = \begin{pmatrix} \boxed{x(t)} \\ y(t) \\ z(t) \end{pmatrix}$$

$y = x'$

$z = x''$

$$X(0) = \begin{pmatrix} u_0 + v_0 + w_0 \\ u_0 - v_0 + 2w_0 \\ u_0 + v_0 + 4w_0 \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\begin{cases} x_1 = \dfrac{2a+b-c}{2} \\ x_2 = \dfrac{2a-3b+c}{6} \quad (1) \\ x_3 = \dfrac{-a+c}{3} \end{cases}$$

$$u_0 = \frac{2x_0 + y_0 - z_0}{2}$$

$$v_0 = \frac{2x_0 - 3y_0 + z_0}{6}$$

$$w_0 = -\frac{x_0 + z_0}{3}$$

$$u_0 e^{t} + v_0 e^{-t} + w_0 e^{2t}$$

$$X(t) = \frac{2x_0 + y_0 - z_0}{2} e^{t} + \frac{2x_0 - 3y_0 + z_0}{6} e^{-t} - \frac{x_0 + z_0}{3} e^{2t}$$

$$x''' - 2x'' - x' + 2x = 0$$

Importante: La solución a la ecuación original es solo la primera entrada