Práctico 9 :

Recordar : • $f: \mathbb{R}^2 \to \mathbb{R}$, $p \in \mathbb{R}^2$ $\Rightarrow$ $D_p^n f(v_1,v_2) = \sum_{j_1+j_2=n} \binom{n}{j_1,j_2} \frac{\partial^n f}{\partial x^{j_1} \partial y^{j_2}}(p) \cdot v_1^{j_1} v_2^{j_2}$

f de clase C^n

$$\binom{n}{j_1,j_2} = \frac{n!}{j_1!\, j_2!}$$

• **Taylor :** $f(p+v) = f(p) + D_p f(v) + \dots + \frac{D_p^n f(v)}{n!} + r_n(v)$

4). Calcular polinomio Taylor de orden 3 de $f(x,y,z) = \frac{yz}{x}$ en el pto $(1,0,0)$.

Evaluamos las derivadas en $(1,0,0)$

- $\frac{\partial f}{\partial x}(x,y,z) = \frac{-yz}{x^2}$ (0)
- $\frac{\partial f}{\partial y}(x,y,z) = \frac{z}{x}$ (0)
- $\frac{\partial f}{\partial z}(x,y,z) = \frac{y}{x}$ (0)
- $\frac{\partial^2 f}{\partial^2 x}(x,y,z) = \frac{1}{2} yz \cdot \frac{1}{x^3}$ (0)
- $\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = -\frac{z}{x^2}$ (0)
- $\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = \frac{-y}{x^2}$ (0)
- $\frac{\partial^2 f}{\partial^2 y} = 0 = \frac{\partial^2 f}{\partial^2 z}$ (0)
- $\frac{\partial^2 f}{\partial z \partial y} = \frac{1}{x}$ (1)

- $\frac{\partial^3 f}{\partial x^3}(x,y,z) = -\frac{1}{2} \cdot \frac{1}{3} yz \frac{1}{x^4}$ (0)
- $\frac{\partial^3 f}{\partial y \partial x^2}(x,y,z) = \frac{1}{2} z \frac{1}{x^3}$ (0)
- $\frac{\partial^3 f}{\partial z \partial x^2}(x,y,z) = \frac{1}{2} y \frac{1}{x^3}$ (0)
- $\overset{(0)}{0} = \frac{\partial^3 f}{\partial^3 z} = \frac{\partial^3 f}{\partial^3 y} = \frac{\partial^3 f}{\partial x \partial^2 z} = \frac{\partial^3 f}{\partial y \partial^2 z} = \frac{\partial^3 f}{\partial z \partial y^2} = \frac{\partial^3 f}{\partial x \partial y^2}$
- $\frac{\partial^3 f}{\partial x \partial y \partial z}(x,y,z) = \frac{-1}{x^2}$ (-1)

$$f(p+v) = f(p) + D_p f(v_1,v_2,v_3) + \frac{D_p^2 f(v_1,v_2,v_3)}{2} + \frac{D_p^3 f(v_1,v_2,v_3)}{6} + r_3(v_1,v_2,v_3)$$

$p+v = (x,y,z)$, $p = (1,0,0)$

$$f(x,y,z) = f(1,0,0) + D_{(1,0,0)} f(x-1,y,z) + \frac{D^2_{(1,0,0)} f(x-1,y,z)}{2} + \frac{D^3_{(1,0,0)} f(x-1,y,z)}{6} + r_3$$

- $D_{(1,0,0)} f(x-1,y,z) = \frac{df}{dx}(1,0,0).(x-1) + \frac{df}{dy}(1,0,0)(y) + \frac{df}{dz}(1,0,0)(z) = 0.$

- $D^2_{(1,0,0)} f(x-1,y,z) = \cancel{\frac{d^2f}{d^2x}(1,0,0)}.(x-1)^2 + \cancel{\frac{d^2f}{dy^2}(1,0,0)}.y^2 + \cancel{\frac{d^2f}{dz^2}(1,0,0)}\, z^2$

$+ 2\cancel{\frac{d^2f}{dxdy}(1,0,0)}(x-1)y. + 2\cancel{\frac{d^2f}{dxdz}(1,0,0)}(x-1)z + 2\frac{d^2f}{dydz}(1,0,0)\, yz$

$= 2.yz.$

- $D^3_{(1,0,0)} f(x-1,y,z) = \underbrace{\ldots\ldots}_{0} + 6\frac{d^3f}{dxdydz}(1,0,0)\,(x-1)yz$

$= -6\,(x-1)yz$

$$f(x,y,z) = 0 + 0 + \frac{2yz}{2} + \frac{(-6)(x-1)yz}{6} + r_3(x-1,y,z)$$

$$\boxed{f(x,y,z) = \underbrace{yz - (x-1)yz}_{\text{polinomio de taylor de orden 3}} + r_3(x-1,y,z)}$$

Obs: Esto se puede calcular más rápido así:

$f(x,y,z) = \frac{yz}{x} = \underbrace{\frac{1}{x}}_{g(x)}.\underbrace{y}_{h(y)}.\underbrace{z}_{i(z)}.$ Para calcular taylor de f alcanza con calcular taylor de orden 3 de $g, h, e\ i$.

Observando que $\frac{1}{x} = 1 - (x-1) + r_2(x)$

entonces $\frac{yz}{x} = \frac{1}{x}.y.z = yz - yz(x-1) + \overbrace{\text{cosas}}^{r_3(x,y,z)}.$

3) b) $f(x,y) = e^x\cos(y)$ (Taylor en $(0,0)$)

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
- $\cos(y) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots$

$\downarrow$

$f(x,y) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + r_3(x)\right)\left(1 - \frac{y^2}{2} + r_3(y)\right)$

$= \underbrace{1 - \frac{y^2}{2} + x - \frac{xy^2}{2} + \frac{x^2}{2} + \frac{x^3}{6}}_{\text{Pol Taylor de orden 3}} + r_3(x,y).$

c) $f(x,y) = \log(1 + \overbrace{xy}^{u})$ (Taylor en $(0,0)$).

• $\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} \dots$

$f(x,y) = xy - \overbrace{\frac{(xy)^2}{2} + r_2(xy)}$

$f(x,y) = \underbrace{xy}_{\text{pol Taylor orden 3.}} + r_3(x,y)$

a) $f(x,y) = \arctan\left(\overbrace{\frac{y}{x^2+1}}^{u}\right)$

• $\operatorname{arctg}(u) = u - \frac{u^3}{3} + \frac{u^5}{5} \dots$

$u = y \cdot \left(\frac{1}{x^2+1}\right) \Rightarrow$ Taylor de u = $y \cdot$ (Taylor de $\frac{1}{x^2+1}$)

· Para calcular taylor orden 3 de f calculo Taylors de $\operatorname{arctg}(u)$ y de $\frac{y}{x^2+1}$.
El Taylor de $\frac{y}{x^2+1}$ "es el producto de los Taylor" de y y $\frac{1}{x^2+1}$.

5) Desarrollar xyz^2 en potencias de x, $(y-1)$ y $(z+1)$.

$$f(p+v) = f(p) + D_p f(v) + \frac{D_p^2 f(v)}{2} + \dots$$

$$f(x,y,z) = f(p) + D_p f(x-p_1, y-p_2, z-p_3) + \frac{D_p^2 f(x-p_1, y-p_2, z-p_3)}{2} + \dots$$

Si $(p_1,p_2,p_3) = (0,1,-1)$ lo de arriba queda escrito como sumas y productos de potencias de x, $(y-1)$ y $(z+1)$.

Como $f(x,y,z) = xyz^2$ es un polinomio de orden 4, su polinomio de Taylor de orden 4 centrado en cualquier lugar coincide con f.

Entonces basta con conseguir Taylor de orden 4 de $f(x,y,z) = xyz^2$ centrado en $(0,1,-1)$

Para esto podemos usar ejercicio 1a).

- Pol Taylor de x centrado en 0 : x.
- pol Taylor de y centrado en 1 : $y = 1 + (y-1)$
- pol Taylor de z^2 centrado en -1 : Si $g(z) = z^2$, $g'(z) = 2z$, $g''(z) = 2$

$$z^2 = g(z) = g(-1) + g'(-1)(z+1) + \frac{g''(-1)(z+1)^2}{2}$$

$$z^2 = 1 + (-2)(z+1) + (z+1)^2$$

Luego $f(x,y,z) = xyz^2 = x.[1 + (y-1)].[1 - 2(z+1) + (z+1)^2].$

6). Calcular pol. Taylor de orden n:

a) $f(x,y,z) = e^{x+y+z}$, en el pto $(0,0,0)$.

Observar que todas las derivadas parciales, de cualquier orden, coinciden con f.

$$D^k_{(0,0,0)} f(x,y,z) = \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} \boxed{\frac{\partial^k f}{\partial x^{j_1} \partial y^{j_2} \partial z^{j_3}}(0,0,0)}_{=1} x^{j_1} y^{j_2} z^{j_3}$$

$$D^k_{(0,0,0)} f(x,y,z) = \sum_{j_1+j_2+j_3=k} \left(\frac{k!}{j_1!\,j_2!\,j_3!}\right) x^{j_1} y^{j_2} z^{j_3}$$

c) $f(x,y) = \frac{1}{xy} = \frac{1}{x}.\frac{1}{y}$ en pto $(1,1)$.

Recordar

$$\frac{1}{1-r} = \sum_0^\infty r^i = 1 + r + r^2 + r^3 + r^4 + \dots$$

$x = 1-r$

$r = 1-x$

$$\frac{1}{x} = \sum_0^\infty (1-x)^i = 1 + (1-x) + (1-x)^2 + \dots$$

$$\frac{1}{x} = 1 + (-1)(x-1) + (-1)^2(x-1)^2 + (-1)^3(x-1)^3 + \dots$$

$\Rightarrow$

$$\frac{1}{x} = \sum_0^n (-1)^i (x-1)^i + r_n(x).$$

$$\frac{1}{y} = \sum_0^n (-1)^i (y-1)^i + r_n(y)$$