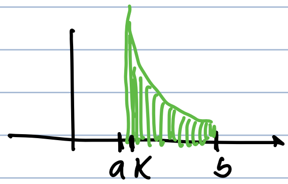
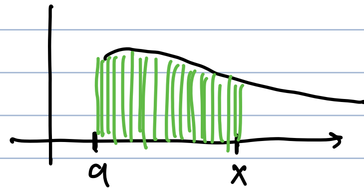
Práctico 5 (impropias):

- **Primera especie**: "generalizar series", f integrable en $[a,\infty)$, definimos
$$\int_a^\infty f := \lim_{K\to\infty} \int_a^K f . \quad \left(\Sigma \sim \int ,\ a_n = f(x) \right)$$

- **Segunda especie**: $f:(a,b]\to\mathbb{R}$, definimos $\int_a^b f = \lim_{K\to a^+} \int_K^b f.$



1) . a) $f:[a,\infty)\to\mathbb{R}^+$ continua. $F(x) = \int_a^x f(t)\,dt$.



Veamos que F es creciente y que $\int_a^\infty f$ converge sii F acot. sup. *

(Un argumento rápido para ver que F es creciente es observar que $F'(x) \geq 0$ $\forall x\in[a,\infty)$
(por TFC $F'(x) = f(x) \geq 0$).)

* Traducción: $\exists \lim_{x\to\infty} F(x)$ sii F acot sup. (Sabiendo F creciente).

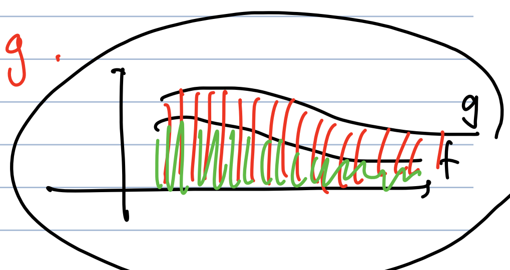
Esto es cálculo 1 y un buen ejercicio para repasar (está escrito bien en soluciones)

b) Criterio de comparación: $0 \leq f \leq g$ continuas en $[a,\infty)$

$$\Rightarrow \int_a^\infty f = \infty \Rightarrow \int_a^\infty g = \infty$$

$$\Rightarrow \int_a^\infty g < \infty \Rightarrow \int_a^\infty f < \infty .$$

Para probar esto definir $F(x) = \int_a^x f$, $G(x) = \int_a^x g$.



2) a) $\displaystyle\int_2^\infty \frac{1}{x\log^\alpha(x)}\,dx$.

Si $u=\log(x)$, $u' = 1/x$

$$\int_2^\infty \frac{1}{x\log^\alpha(x)}\,dx \longrightarrow = \int_{u(2)}^{\text{"}u(\infty)\text{"}} \frac{1}{u^\alpha}\, u'dx \; = \int_{\log(2)}^{\infty} \frac{1}{u^\alpha}\,du$$

$du = u'dx$.

Formalmente :

$$\int_2^\infty \frac{1}{x}\cdot\frac{1}{(\log(x))^\alpha}\,dx = \lim_{K\to\infty}\int_2^K \frac{1}{x}\,\frac{1}{(\log(x))^\alpha}\,dx = \lim_{K\to\infty}\int_{\log(2)}^{\log(K)} \frac{1}{u^\alpha}\,du$$

c. variable.

$$\int \frac{1}{u^\alpha}\,du = \int u^{-\alpha}\,du \xrightarrow{\alpha\neq 1} \frac{u^{-\alpha+1}}{-\alpha+1} = \frac{1}{1-\alpha}\cdot\frac{1}{u^{\alpha-1}}$$

$$\xrightarrow{\alpha=1} \log(u).$$

$$\int_{\log(2)}^{\log(K)} \frac{1}{u^\alpha} \xrightarrow{\alpha\neq 1} \frac{1}{1-\alpha}\cdot\frac{1}{u^{\alpha-1}}\Bigg|_{\log(2)}^{\log(K)} = \frac{1}{1-\alpha}\left(\frac{1}{\log(K)^{\alpha-1}} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

$$\xrightarrow{\alpha=1} \log(u)\Bigg|_{\log(2)}^{\log(K)} = \log(\log(K)) - \log(\log(2)).$$

Tomando lim en K :

Si $\alpha = 1$,
$$\int_2^\infty \frac{1}{x}\,\frac{1}{(\log(x))^\alpha}\,dx = \lim_{K\to\infty} \log(\log(K)) - \log(\log(2))$$
$$= \infty.$$

Si $\alpha\neq 1$
$$\int_2^\infty \frac{1}{x}\,\frac{1}{(\log(x))^\alpha}\,dx = \lim_{K\to\infty} \frac{1}{1-\alpha}\left(\frac{1}{\log(K)^{\alpha-1}} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

• obs: Si $\alpha-1>0$ $(\alpha>1)$ $\Rightarrow$ $\log(K)^{\alpha-1} \xrightarrow{K\to\infty} \infty$

Si $\alpha-1<0$ $(\alpha<1)$ $\Rightarrow$ $\log(K)^{\alpha-1} = \dfrac{1}{\log(K)^{1-\alpha}} \xrightarrow{K\to\infty} 0$

Si $\alpha\neq 1$,
$$\int_2^\infty \frac{1}{x}\,\frac{1}{(\log(x))^\alpha}\,dx = \begin{cases} \infty & \text{si } \alpha<1 \\ \boxed{\dfrac{1}{1-\alpha}\left(\dfrac{-1}{\log(2)^{\alpha-1}}\right)} & \text{si } \alpha>1 \end{cases}$$

En resumen diverge para $\alpha\leq 1$, converge $\alpha>1$

e) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{e^{x}+e^{-x}}\,dx}_{I} + \underbrace{\int_{0}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx}_{I}$ (son iguales).

Si $u=-x$ $\displaystyle\int_{u(-\infty)}^{u(0)} (-1)\frac{1}{e^{-u}+e^{u}}\overbrace{(-1)\,dx}^{du} = -1\int_{+\infty}^{0} \frac{1}{e^{u}+e^{-u}}\,du = \int_{0}^{\infty} \frac{1}{e^{u}+e^{-u}}\,du.$

$du = u'dx$

En gral, si f es par $\displaystyle\int_{0}^{K} f = \int_{-K}^{0} f.$ Nuestra f es par $\left(f(x) = \frac{1}{e^{x}+e^{-x}}\right)$

Prueba: $f(-x) = \dfrac{1}{e^{-x}+e^{-(-x)}} = \dfrac{1}{e^{-x}+e^{x}} = f(x)$

Entonces $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx = 2\int_{0}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx. = 2I.$

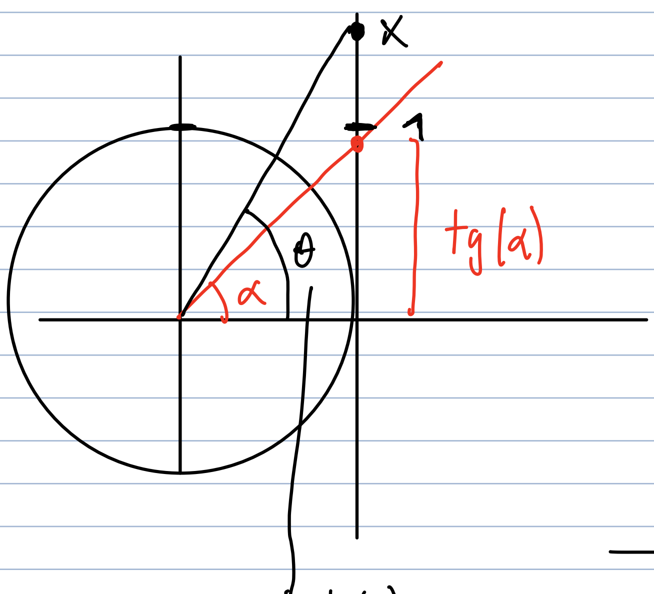
$$I = \int_{0}^{\infty} \frac{1}{e^{x}+e^{-x}}\,dx = \lim_{K\to\infty} \int_{0}^{K} \frac{1}{e^{x}+e^{-x}}\,dx. = \lim_{K\to\infty} \int_{0}^{K} \frac{e^{x}\,dx}{(e^{x})^2+1}$$

$u = e^{x}$, $u' = e^{x}$, $du = e^{x}\,dx$ $\longrightarrow$ $\displaystyle = \lim_{K\to\infty} \int_{1}^{e^{K}} \frac{1}{u^2+1}\,du.$

$$= \lim_{K\to\infty} \operatorname{arctg}(u)\Big|_{1}^{e^{K}}.$$

$$= \operatorname{arctg}(\infty) - \operatorname{arctg}(1).$$

$$= \pi/2 - \pi/4. = \pi/4.$$



$\theta = \operatorname{arctg}(x)$.

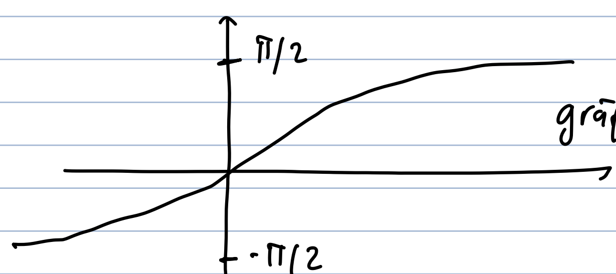


gráfico arctg(x).

d) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\ \operatorname{arctg}(x)\Big|_{0}^{\infty} = 2 \cdot \left(\frac{\pi}{2}\right) = \pi.$