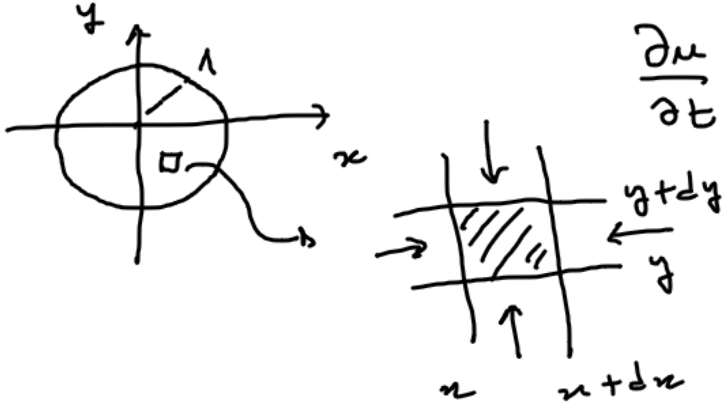
ECUACIÓN DE LAPLACE (en el disco)

Motivación:

1) La ecuación del calor para una barra $\qquad \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

La ecuación del calor en un disco $u(x,y,t)$



$$\frac{\partial u}{\partial t} = \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\text{Laplaciano}} = \Delta u = \nabla^2 u$$

Ecuación de Laplace.

Solución estacionaria $\dfrac{\partial u}{\partial t} = 0 \implies$ $\boxed{\Delta u = 0}$

Funciones armónicas

2) Electromagnetismo: La ley de Gauss $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ ← densidad de carga

↑ campo eléctrico

$E = -\nabla \varphi$ $\quad \varphi$: potencial eléctrico

En una región sin cargas:

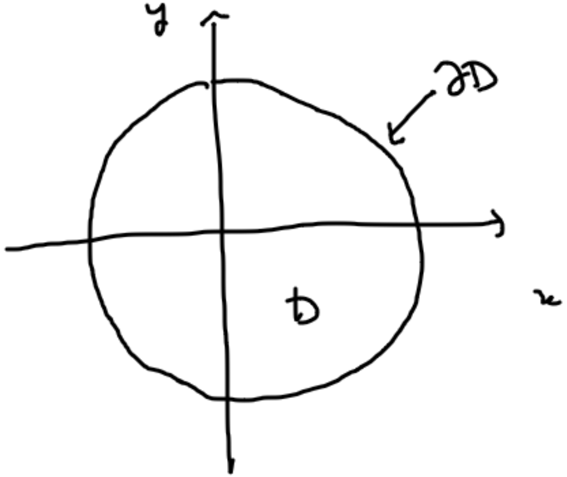
$\nabla \cdot (\nabla \varphi) = 0 \Rightarrow \boxed{\nabla^2 \varphi = 0}$ Ecuación de Laplace.

La ecuación de Laplace en disco $D = \{(x,y) : x^2 + y^2 < 1\}$

es: $\Delta u = 0$, $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ → NO INTERVIENE EL TIEMPO (orden cero)

Para el problema Cauchy-Dirichlet

solo hay condiciones de borde.

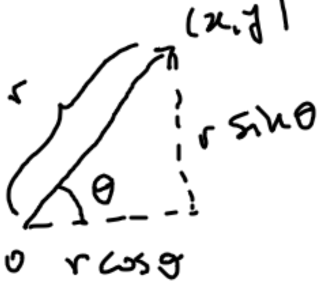


$$\partial D = \{ (x,y) : x^2 + y^2 = 1 \}$$

$$\text{PCD} \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \forall\, (x,y) \in D \\[2ex] u(x,y) = u_0(x,y) & \forall\, (x,y) \in \partial D . \end{cases}$$

LA ECUACIÓN DE LAPLACE EN COORDENADAS POLARES:

$$\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \end{cases} \qquad \begin{cases} r = (x^2+y^2)^{1/2} = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$



$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta} \cdot \frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \sin\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$= \frac{\cos\theta}{r}$$

$$= -\frac{y}{r} \cdot \frac{1}{r} = -\frac{\operatorname{sen}\theta}{r}$$

En resumen:

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{\operatorname{sen}\theta}{r}$$

$$\frac{\partial r}{\partial y} = \operatorname{sen}\theta \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \operatorname{sen}\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \end{aligned}}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}\right)^2 + \left(\operatorname{sen}\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} - \cos\theta \frac{\partial}{\partial r}\frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta} - \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial}{\partial r} + \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}\frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}$$

$$\operatorname{sen}^2\theta \frac{\partial^2}{\partial r^2} + \operatorname{sen}\theta\frac{\partial}{\partial r}\frac{\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\operatorname{sen}\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$= \boxed{\frac{\partial^2}{\partial r^2}} - \cos\theta\operatorname{sen}\theta\left[\cancel{-\frac{1}{r^2}\frac{\partial}{\partial\theta}} + \cancel{\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}}\right] - \frac{\operatorname{sen}\theta}{r}\left[-\operatorname{sen}\theta\frac{\partial}{\partial r} + \cancel{\cos\theta\frac{\partial^2}{\partial\theta\partial r}}\right]$$

$$+ \frac{\operatorname{sen}\theta}{r^2}\left[\cancel{\cos\theta\frac{\partial}{\partial\theta}} + \operatorname{sen}\theta\frac{\partial^2}{\partial\theta^2}\right] + \operatorname{sen}\theta\cos\theta\left[\cancel{-\frac{1}{r^2}\frac{\partial}{\partial\theta}} + \cancel{\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}}\right] + \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial}{\partial r} + \cancel{\operatorname{sen}\theta\frac{\partial^2}{\partial\theta\partial r}}\right]$$

$$+ \frac{\cos\theta}{r^2}\left[\cancel{-\operatorname{sen}\theta\frac{\partial}{\partial\theta}} + \cos\theta\frac{\partial^2}{\partial\theta^2}\right] \quad \Big\| \quad \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \Delta$$

En resumen:

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$$

$u(r,\theta)$ Condición de borde $u(1,\theta) = f(\theta)$

Separación de variables: $u(r,\theta) = R(r)\,T(\theta)$

$$\frac{\partial u}{\partial r} = R'T \qquad \frac{\partial^2 u}{\partial r^2} = R''T \qquad \frac{\partial^2 u}{\partial \theta^2} = RT''$$

$$\boxed{R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0}$$

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}RT'' = 0$$

$$r^2R''T + rR'T + RT'' = 0$$

$$(r^2R'' + rR')T = -RT''$$

$$\boxed{\frac{r^2R'' + rR'}{R} = \frac{-T''}{T}}$$

Obtenemos las ecuaciones:

$$\begin{cases} r^2R'' + rR' = \lambda R & (*_r) \\ T'' = -\lambda T & (*_\theta) \end{cases}$$

$\rightsquigarrow \lambda = n^2, \; n = 1, 2, \ldots$

$(*_\theta)$ $\quad T'' = -\lambda T$

$T(\theta + 2\pi) = T(\theta)$

$$\rightsquigarrow \begin{cases} T(\theta) = A\cos(\sqrt{\lambda}\,\theta) + B\,\mathrm{sen}(\sqrt{\lambda}\,\theta) & \lambda > 0 \\ T(\theta) = A & \lambda = 0 \end{cases}$$

$$\boxed{T_n(\theta) = A_n \cos(n\theta) + B_n \operatorname{sen}(n\theta)}$$

$\lambda = n^2$

$n = 0, 1, 2, \ldots$

$$r^2 R'' + r R' = \lambda R \rightsquigarrow \text{Ecuación de Euler}$$

$$R(r) = r^\alpha \;, \quad R' = \alpha r^{\alpha-1} \;, \quad R'' = \alpha(\alpha-1) r^{\alpha-2}$$

$$\alpha(\alpha-1)\cancel{r^\alpha} + \alpha \cancel{r^\alpha} = \lambda \cancel{r^\alpha}$$

$$\alpha^2 - \alpha + \alpha = \lambda$$

$$\alpha^2 = \lambda$$

$$\alpha = \pm\sqrt{\lambda} = \pm n$$

$$\boxed{\begin{array}{c} n = 1, 2, \ldots \\ r^n, \; r^{-n} \\ R(r) = C_n r^n + D_n r^{-n} \end{array}}$$

$n=0$ $\quad r^2R'' + rR' = 0$

$$r^2R'' = -rR'$$

$$(\ln R')' = \frac{R''}{R'} = -\frac{1}{r}$$

$$\ln R' = K - \ln r$$

$$\ln rR' = K$$

$$rR' = \underbrace{e^K}_{K'}$$

$$\Rightarrow \quad R' = \frac{K'}{r}$$

$$R = \underbrace{K''}_{C_0} + \underbrace{K'}_{D_0}\ln r$$

En conclusión:

$$\boxed{\begin{array}{ll} R_n(r) = C_n r^n + D_n r^{-n} & n \geq 1 \\ R_0(r) = C_0 + D_0 \ln r & n = 0 \end{array}}$$

La solución general para $\Delta u = 0$ en el disco y en coordenadas polares es:

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{n=1}^{\infty} \left(C_n r^n + D_n r^{-n}\right)\left(A_n \cos(n\theta) + B_n \operatorname{sen}(n\theta)\right)$$

Como u es continua en D (y por ende acotada)

$$\Rightarrow D_n = 0 \quad \forall n \geq 0$$

$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \operatorname{sen}(n\theta)}$$

Usamos la condición de borde: $u(1,\theta) = f(\theta)$

$$u(1,\theta) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \operatorname{sen}(n\theta)}_{\underline{\underline{f(\theta)}}}$$

$\Rightarrow$ a_n y b_n son los coef. de Fourier de $f(\theta)$.