LA CUERDA ACOTADA

Consideremos el problema de Cauchy-Dirichlet siguiente:

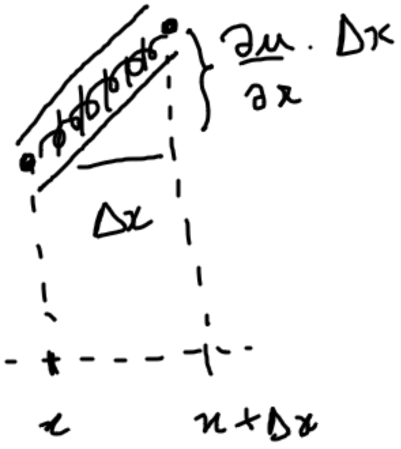
$$
(*)\quad \begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \quad \forall\ (x,t) \in (0,L)\times(-\infty,+\infty) \\[2ex]
\left.\begin{array}{l} u(x,0) = u_0(x) \\[1ex] \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \end{array}\right\} \text{condiciones iniciales} \\[4ex]
\left.\begin{array}{l} u(0,t) = p_1(t) \\[1ex] u(L,t) = p_2(t) \end{array}\right\} \text{condiciones de borde}
\end{cases}
$$

TEOREMA: El problema $(*)$ tiene soluciones únicas

DEM: Sean u_1 y u_2 dos soluciones de $(*)$

Llamamos $u = u_1 - u_2$.

Observar que u es solución del problema:



$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = 0 \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \\ u(0,t) = 0 \leftarrow \\ u(L,t) = 0 \leftarrow \end{cases}$$

Vamos a probar que $u \equiv 0$.

Para esto vamos a probar primero que la energía de la cuerda se conserva.

$$I(t) = \frac{1}{2} \int_0^L \underbrace{\left(\frac{\partial u}{\partial x}\right)^2}_{\text{energía potencial}} + \underbrace{\left(\frac{\partial u}{\partial t}\right)^2 \frac{1}{c^2}}_{\text{energía cinética}} \, dx$$

$$I(t) = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 dx$$

$$I'(t) = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2\right] dx$$

$$= c^2 \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{1}{\cancel{2}}\int_0^L \cancel{2}\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial t\,\partial x} + \frac{\cancel{2}}{\cancel{c^2}}\,\frac{\partial u}{\partial t}\,\boxed{\frac{\partial^2 u}{\partial t^2}}\; dx$$

$$= \int_0^L \frac{\partial^2 u}{\partial x\,\partial t}\cdot\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x^2}\; dx$$

$A = \frac{\partial u}{\partial x}$ $B = \frac{\partial u}{\partial t}$

$$= \int_0^L \frac{\partial}{\partial x}B \cdot A + B\cdot\frac{\partial}{\partial x}A \; dx = \int_0^L \frac{\partial}{\partial x}(AB)\, dx$$

$$I'(t) = \int_0^L \frac{\partial}{\partial x}(AB)\,dx = AB\Big|_0^L = A(L,t)\underbrace{B(L,t)}_{0} - A(0,t)\underbrace{B(0,t)}_{0}$$

$$= 0$$

$$I(t) = \text{constante}$$

Energía en $t=0$, $$I(0) = \frac{1}{2}\int_0^L \Big(\underbrace{\frac{\partial u}{\partial x}(x,0)}_{\substack{\text{pendiente}\\\text{inicial}\\ 0}}\Big)^2 + \Big(\underbrace{\frac{1}{c}\frac{\partial u}{\partial t}(x,0)}_{\substack{\text{velocidad}\\\text{inicial}\\ 0}}\Big)^2 dx = 0$$

$$\Rightarrow I(t) = 0 \quad \forall t$$

$$I(t) = \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 dx = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \quad \frac{\partial u}{\partial t} = 0 \Rightarrow u = cte$$

$$\Rightarrow u \equiv 0.$$

EXISTENCIA DE SOLUCIONES PARA CONDICIONES DE BORDE NULAS:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \\ u(0,t) = u(l,t) = 0 \end{cases}$$

Buscamos soluciones de la forma

$$u_k(x,t) = X_k(x)\, T_k(t) \qquad k \in \mathbb{N}$$

$$X_k T_k'' = c^2 X_k'' T_k$$

solo depende de x $\rightsquigarrow$ $\dfrac{X_k''}{X_k} = \dfrac{1}{c^2} \dfrac{T_k''}{T_k}$ $\leftsquigarrow$ solo depende de t

$\Rightarrow \exists$ cte $\lambda \in \mathbb{R}$ / $\dfrac{X_k''}{X_k} = \lambda$ y $\dfrac{T_k''}{T_k} = c^2 \lambda$

La solución general para X_k es

$$X_k = \begin{cases} a e^{\sqrt{\lambda} x} + b e^{-\sqrt{\lambda} x} & \lambda > 0 \\ a + bx & \lambda = 0 \\ a \cos(\sqrt{-\lambda} x) + b \operatorname{sen}(\sqrt{-\lambda} x) & \underline{\underline{\lambda < 0}} \end{cases}$$

Lo mismo para T_k.

Condición de borde: $X_k(0) T_k(t) = X_k(L) T_k(t) = 0 \quad \forall t$

Si $\mu_k \neq 0 \Rightarrow X_k(0) = X_k(L) = 0$.

$\lambda > 0)$ $\left.\begin{array}{l} a + b = 0 \\ a e^{\sqrt{\lambda} L} + b e^{-\sqrt{\lambda} L} = 0 \end{array}\right\} \Rightarrow a = b = 0$ $\quad$ $\lambda = 0)$ $\left.\begin{array}{l} a = 0 \\ a + bL = 0 \end{array}\right\} \Rightarrow a = b = 0$

$\lambda < 0)$ $\quad X_k(0) = a \underbrace{\cos(\sqrt{-\lambda} \cdot 0)}_{1} + b \underbrace{\operatorname{sen}(\sqrt{-\lambda} \cdot 0)}_{0} = 0$

$$\Rightarrow a = 0$$

$$X_k(L) = b \operatorname{sen}\left(\sqrt{-\lambda}\, L\right) = 0 \Rightarrow \sqrt{-\lambda}\, L = k\pi$$

$$k \in \{1, 2, 3, \ldots\}$$

$$\Rightarrow \lambda = -\left(\frac{k\pi}{L}\right)^2 \qquad X_k(x) = \underbrace{(cte)}_{1} \operatorname{sen}\left(\frac{k\pi}{L} x\right)$$

(lo elegimos así)

Para T_k obtenemos $T_k(t) = a_k \cos\left(\frac{ck\pi}{L} t\right) + b_k \operatorname{sen}\left(\frac{ck\pi}{L} t\right)$

$$u_k(x,t) = \operatorname{sen}\left(\frac{k\pi}{L} x\right)\left[a_k \cos\left(\frac{ck\pi}{L} t\right) + b_k \operatorname{sen}\left(\frac{ck\pi}{L} t\right)\right]$$

Las soluciones $u_k(x,t) = \text{sen}\left(\frac{k\pi}{L}x\right)\left[a_k \cos\left(\frac{ck\pi}{L}t\right) + b_k \text{sen}\left(\frac{ck\pi}{L}t\right)\right]$

se llaman modos normales.

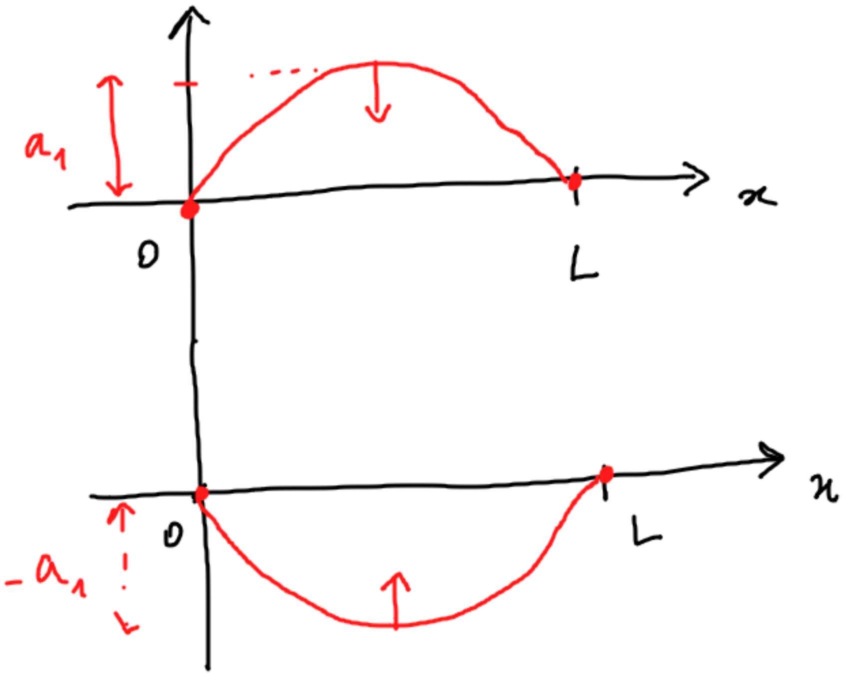
$k=1$ $\quad t=0 \quad u_1(x,0) = a_1 \text{sen}\left(\frac{\pi}{L}x\right)$

$$\frac{ck\pi}{L}(t+\tau) = \frac{ck\pi}{L}t + 2\pi$$

$$\frac{ck\cancel{\pi}}{L}\tau = 2\cancel{\pi}$$

$$\tau = \frac{2L}{ck}$$

$\tau = \frac{2L}{c}$



$t = \tau/2$

$t = L/c$

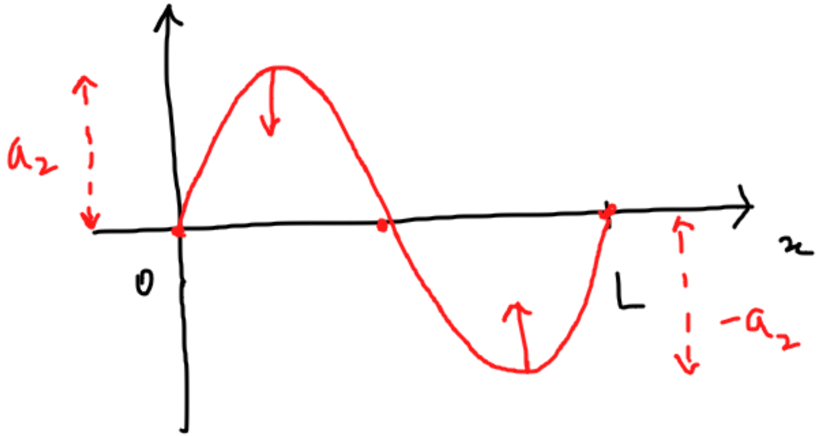
$\lambda\nu = c$

$$\nu = \frac{ck}{2L}$$

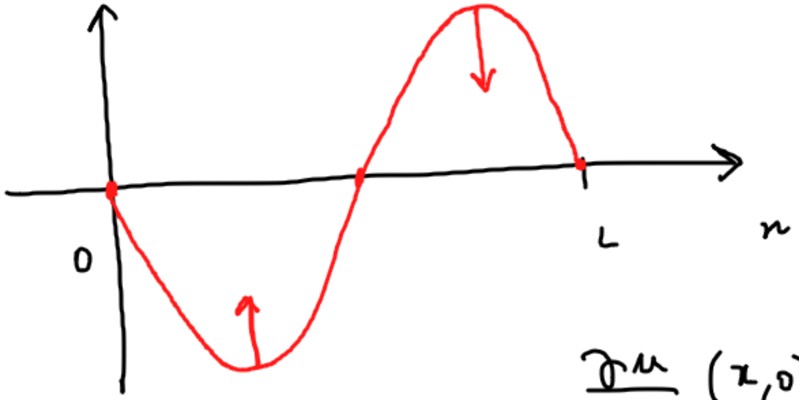
$$\lambda = \frac{2L}{k}$$

$k=2$ $$u_2(x,t) = \operatorname{sen}\left(\frac{2\pi}{L}x\right)\left[a_2\cos\left(\frac{2c\pi}{L}t\right)+b_2\operatorname{sen}\left(\frac{2c\pi}{L}t\right)\right]$$

$t=0$



$t=T/2$



La solución general es de la forma:

$$u = \sum_{k=1}^{\infty} u_k$$

CONDICIONES INICIALES

$$u(x,0) = \sum_{k=1}^{\infty} a_k \operatorname{sen}\left(\frac{k\pi}{L}x\right) = u_0(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} b_k \frac{k\pi c}{L}\operatorname{sen}\left(\frac{k\pi}{L}x\right) = v_0(x)$$

En conclusión: - los $\boxed{a_k}$ son los coef. de Fourier en senos de $u_0(x)$.

- los $\boxed{b_k \frac{k\pi c}{L}}$ son los coef. de Fourier en senos de $v_0(x)$.

EXISTENCIA DE SOLUCIONES

Si $\{a_k\}_{k=1}^{\infty}$ y $\{b_k\}_{k=1}^{\infty}$ satisfacen $\boxed{|a_k| \leq \frac{C}{k^4}, \; |b_k| \leq \frac{C}{k^4}}$

entonces la función

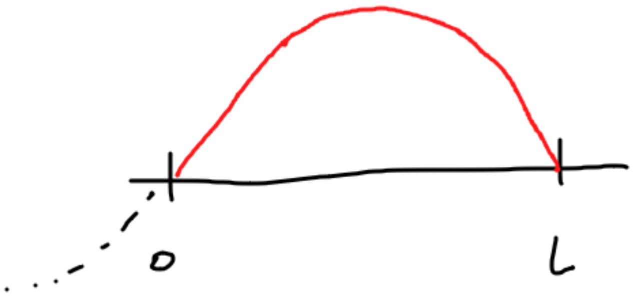
$$u(x,t) = \sum_{k=1}^{\infty} \operatorname{sen}\left(\frac{k\pi}{L}x\right)\left[a_k \cos\left(\frac{ck\pi}{L}t\right) + b_k \operatorname{sen}\left(\frac{ck\pi}{L}t\right)\right]$$

es solución al problema:

- u es C^2 en $(0,L)\times\mathbb{R}$ y continua en $[0,L]\times\mathbb{R}$
- $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
- $u(0,t) = u(L,t) = 0$
- $u(x,0) = \sum_{k=1}^{\infty} a_k \operatorname{sen}\left(\frac{k\pi}{L}x\right)$ y $\frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} \frac{ck\pi}{L} b_k \operatorname{sen}\left(\frac{k\pi}{L}x\right)$

"Ejemplo:"

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = \underbrace{x(L-x)}_{u_0(x)} \qquad \dfrac{\partial u}{\partial t}(x,0) = \underbrace{0}_{v_0(x)} \end{cases}$$



- $f(x)$ = la extensión impar $2L$-periódica de $u_0(x)$
- Calculamos $a_k = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,\mathrm{sen}\left(\dfrac{k\pi}{L}x\right)dx$

- Hacemos lo mismo con $v_0(x)$

$b_n = 0$