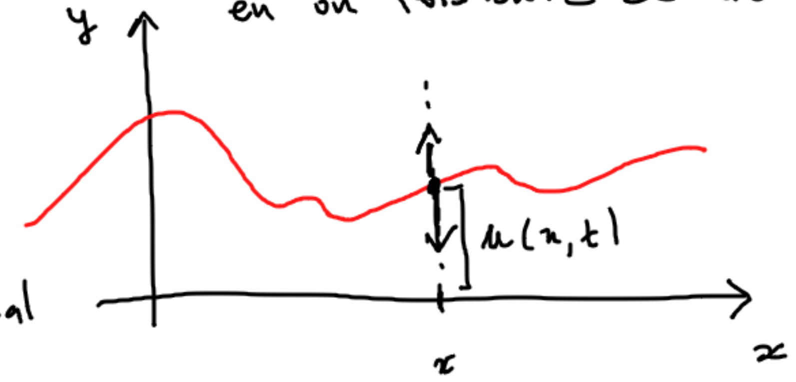
# Ecuación de ondas (en una cuerda)

El movimiento se produce en un plano $(x,y)\in\mathbb{R}^2$

$x$ = posición horizontal $\quad y$ = posición vertical

en un instante de tiempo $t$

Cuerda infinita




$t$ = tiempo

Asumimos un movimiento vertical

$u(x,t)$ = posición vertical de la cuerda sobre el punto $x$ en tiempo $t$.

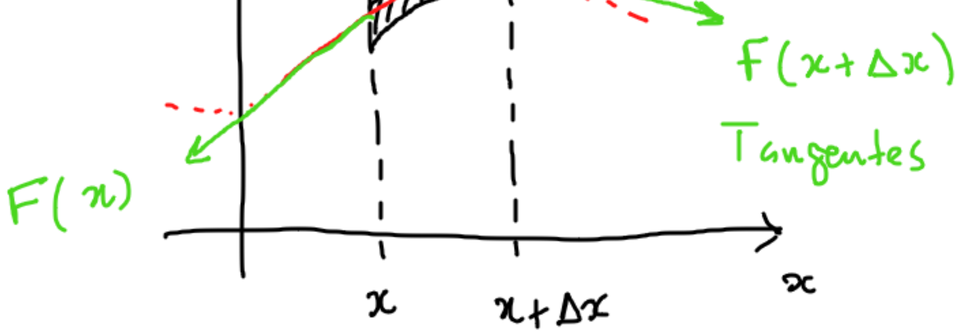
$\dfrac{\partial u}{\partial t}(x,t)$ = velocidad de la cuerda sobre el punto $x$ en tiempo $t$

$\dfrac{\partial^2 u}{\partial t^2}(x,t)$ = aceleración

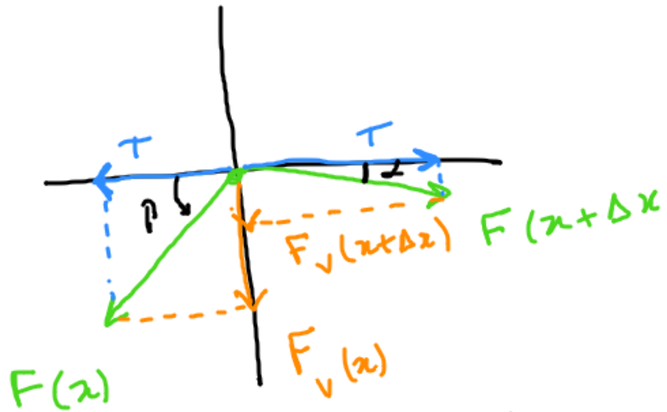
$\rho$ = densidad de masa de la cuerda uniforme

$T$ = tensión de la cuerda uniforme

Masa $= \rho \Delta x$




La tangente a la cuerda viene dada por $\dfrac{\partial u}{\partial x}$




$\Delta x$ muy chico

$$\frac{F_v(x)}{T} = \tan(\beta) = \frac{\partial u}{\partial x}(x)$$


$$\frac{F_v(x+\Delta x)}{T} = \tan(\alpha) = \frac{\partial u}{\partial x}(x+\Delta x)$$

$$\rho \Delta x \frac{\partial^2 u}{\partial t^2}(x) = F_v(x+\Delta x) - F_v(x) = T\left(\frac{\partial u}{\partial x}(x+\Delta x) - \frac{\partial u}{\partial x}(x)\right)$$

$$\frac{\partial^2 u}{\partial t^2}(x)\Big/_t = \left(\frac{T}{\rho}\right) \frac{\frac{\partial u}{\partial x}(x+\Delta x)\Big/_t - \frac{\partial u}{\partial x}(x)\Big/_t}{\Delta x}$$

Haciendo $\Delta x \to 0$ :

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \left(\frac{T}{\rho}\right) \frac{\partial^2 u}{\partial x^2}}$$

Si llamamos $c^2 = \dfrac{T}{\rho}$ que $\boxed{\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$

$c$ determina la <u>velocidad</u> de propagación

$$[c] = \left[\frac{T}{\rho}\right]^{1/2} = \left[\frac{N}{kg/m}\right]^{1/2} = \left[\frac{\cancel{kg}\, m/s^2}{\cancel{kg}/m}\right]^{1/2} = \left[\frac{m^2}{s^2}\right]^{1/2} = \left[m/s\right]$$

- Ecuación de ondas $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

- Condiciones iniciales: $\left\{ \begin{array}{ll} \text{posición inicial} & u_0(x) = u(x,0) \\ \text{velocidad inicial} & v_0(x) = \dfrac{\partial u}{\partial t}(x,0) \end{array} \right.$

  2do orden

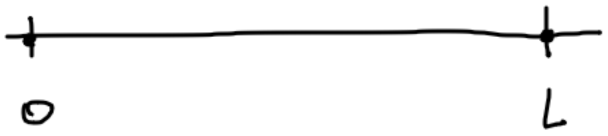
- Condición de borde: → cuerda infinita sin borde
  → cuerda acotada

PROBLEMA DE CAUCHY-DIRICHLET PARA LA CUERDA INFINITA

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \quad \text{en} \quad \mathbb{R}\times\mathbb{R} \\ u(x,0) = u_0(x) \\ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \end{array} \right.$$

# PROBLEMA DE CAUCHY-DIRICHLET PARA LA CUERDA ACOTADA

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \\[2ex]
\left.\begin{array}{l} u(x,0) = u_0(x) \\[1ex] \dfrac{\partial u}{\partial t}(x,0) = v_0(x) \end{array}\right\} \text{INICIAL} \\[4ex]
\left.\begin{array}{l} u(0,t) = p_1(t) \\[1ex] u(L,t) = p_2(t) \end{array}\right\} \text{BORDE}
\end{cases}
$$



# MÉTODO DE PROPAGACIÓN PARA LA CUERDA INFINITA

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Cambio de variable $\xi = x - ct \quad \eta = x + ct$

$$x = \frac{\xi + \eta}{2} \qquad t = \frac{\eta - \xi}{2c}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial u}{\partial \eta} \cancelto{1}{\frac{\partial \eta}{\partial x}}$$

(regla de la cadena)

$$= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} \cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial^2 u}{\partial \eta \partial \xi} \cancelto{1}{\frac{\partial \eta}{\partial x}} + \frac{\partial^2 u}{\partial \xi \partial \eta} \cancelto{1}{\frac{\partial \xi}{\partial x}} + \frac{\partial^2 u}{\partial \eta^2} \cancelto{1}{\frac{\partial \eta}{\partial x}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cancelto{-c}{\frac{\partial \xi}{\partial t}} + \frac{\partial u}{\partial \eta}\cancelto{c}{\frac{\partial \eta}{\partial t}}$$

$\xi = x - ct \quad \eta = x + ct$

$$= -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

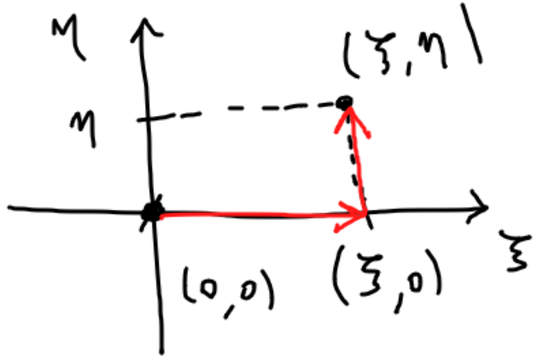
$$\frac{\partial^2 u}{\partial t^2} = -c\frac{\partial^2 u}{\partial \xi^2}\cancelto{-c}{\frac{\partial \xi}{\partial t}} - c\frac{\partial^2 u}{\partial \xi \partial \eta}\cancelto{c}{\frac{\partial \eta}{\partial t}} + c\frac{\partial^2 u}{\partial \xi \partial \eta}\cancelto{-c}{\frac{\partial \xi}{\partial t}} + c\frac{\partial^2 u}{\partial \eta^2}\cancelto{c}{\frac{\partial \eta}{\partial t}}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2}} = \cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2\frac{\partial^2 u}{\partial \eta^2}} = \boxed{c^2\frac{\partial^2 u}{\partial x^2}}$$

$$= \cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} + 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2\frac{\partial^2 u}{\partial \eta^2}} \Rightarrow \boxed{\frac{\partial^2 u}{\partial \xi \partial \eta} = 0}$$

ECUACIÓN DE ONDAS EN $\xi$ Y $\eta$

$$\boxed{\frac{\partial^2 u}{\partial\xi\,\partial\eta} = 0}$$




$$\frac{\partial}{\partial\xi}\left[\frac{\partial u}{\partial\eta}\right] = 0 \Rightarrow \frac{\partial u}{\partial\eta} = g(\eta)$$

NO DEPENDE DE $\xi$

$$\frac{\partial}{\partial\eta}\left[\frac{\partial u}{\partial\xi}\right] = 0 \Rightarrow \frac{\partial u}{\partial\xi} = f(\xi)$$

NO DEPENDE DE $\eta$

$$u(\xi,\eta) - \overbrace{u(0,0)}^{cte}$$

$$= \int_0^{\xi} \frac{\partial u}{\partial\xi} + \int_0^{\eta} \frac{\partial u}{\partial\eta}$$

$$= F(\xi) + G(\eta)$$

$$\Rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$$

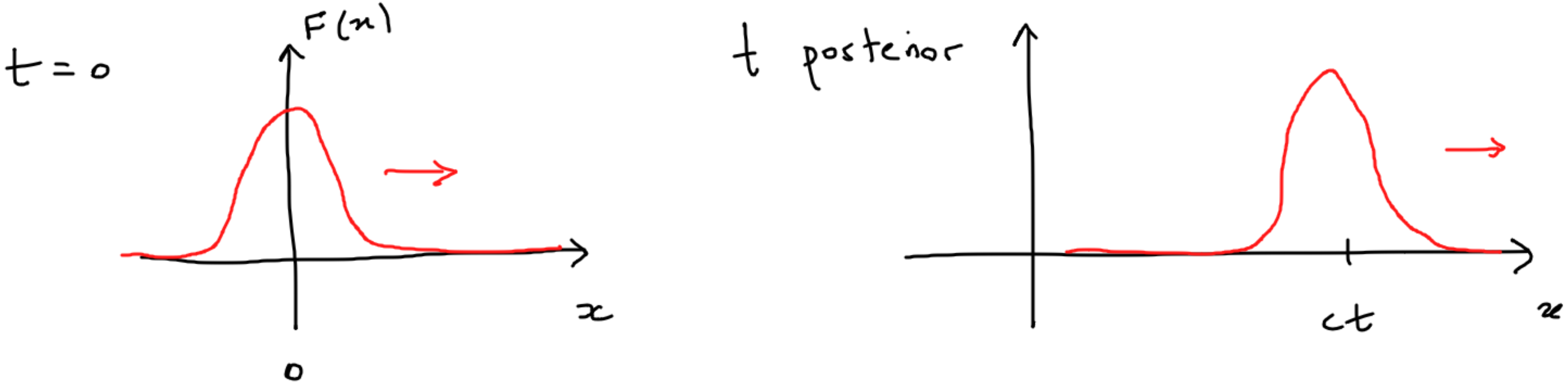
$$u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\frac{\partial u}{\partial \xi} = F'(\xi) \longrightarrow \frac{\partial^2 u}{\partial \eta \partial \xi} = 0$$

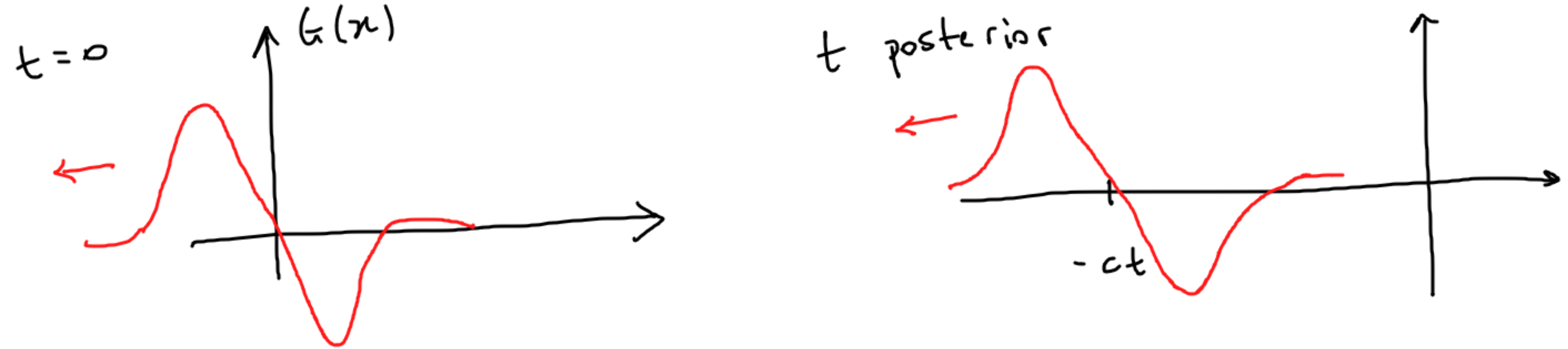
**Resumen**: $\boxed{\text{Las soluciones de } \dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0 \text{ son todas de la forma } u(\xi,\eta) = F(\xi) + G(\eta)}$

Volviendo a $x$ y $t$: $\boxed{\text{Las soluciones de } \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \text{ son de la forma } u(x,t) = F(x-ct) + G(x+ct)}$

$F(x-ct)$ onda viajera hacia la derecha con velocidad $c$




$G(x+ct)$ onda viajera hacia la izquierda con velocidad $c$



Calculamos $F$ y $G$ a partir de las condiciones iniciales

$$u(x,t) = F(x-ct) + G(x+ct)$$

Posición inicial: $u(x,0) = \underline{F(x) + G(x) = u_0(x)}$

Velocidad inicial: $\dfrac{\partial u}{\partial t}(x,0) = \Big[ -F'(x-ct)c + G'(x+ct)c \Big]_{t=0}$

$$= -cF'(x) + c\,G'(x)$$

$$= c\left[ G'(x) - F'(x) \right] = v_0(x)$$

Si integramos

$$\int_0^x v_0(y)\,dy = c\int_0^x G'(y) - F'(y)\,dy = \underline{c\big(G(x)-F(x)\big)}\ \overbrace{-c\big(G(0)-F(0)\big)}^{K}$$

$$\begin{cases} F(x) + G(x) = u_0(x) \\ G(x) - F(x) = \dfrac{1}{c}\displaystyle\int_0^x v_0(y)\,dy + K/c \end{cases}$$

$$\Rightarrow \begin{cases} G(x) = \dfrac{1}{2}u_0(x) + \dfrac{1}{2c}\displaystyle\int_0^x v_0(y)\,dy + K/2c \\ F(x) = 1/2\, u_0(x) - \dfrac{1}{2c}\displaystyle\int_0^x v_0(y)\,dy - K/2c \end{cases}$$

$$u(x,t) = F(x-ct) + G(x+ct) = 1/2\, u_0(x-ct) - \frac{1}{2c}\int_0^{x-ct} v_0(y)\,dy - \cancel{K/2c}$$

$$+ 1/2\, u_0(x+ct) + 1/2c \int_0^{x+ct} v_0(y)\,dy + \cancel{K/2c}$$

$$\boxed{u(x,t) = \frac{u_0(x-ct) + u_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)\,dy}$$

Observar que la solución al problema de Cauchy-Dirichlet para la cuerda infinita es unica.