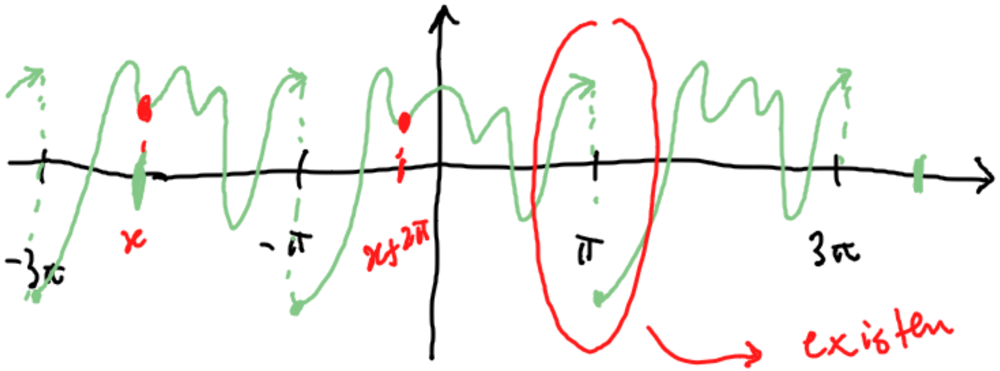
Series de Fourier

Espacio vectorial $V = \{ f : \mathbb{R} \to \mathbb{R} : \underbrace{\text{continua a trozos}}_{2\pi-\text{periódicas}} \}$

(Mas adelante veremos cualquier período)

Producto interno (escalar): $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x)\, dx$

Las discontinuidades son finitas en un intervalo, y además son de tipo salto.



$f(x) = f(x+2\pi)$

existen los límites por izq. y por der.

"La base" de senos y cosenos:

$$S = \{ 1/\sqrt{2}, \operatorname{sen}(x), \cos(x), \operatorname{sen}(2x), \cos(2x), \dots \} = \{ 1/\sqrt{2} \} \cup \{ \operatorname{sen}(nx), \cos(nx) \}_{n \geq 1}$$

Propiedad básica: S es un conjunto ortonormal:

1) Si $f \neq g \in S \Rightarrow \langle f, g \rangle = 0$

2) $\|f\| = 1 \quad \forall f \in S$

$\|f\|^2 = \langle f, f \rangle$

$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx$

Dem:

2) $$\left\| \frac{1}{\sqrt{2}} \right\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2}} \right)^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = \frac{2\pi}{2\pi} = 1$$

$$\| \operatorname{sen}(nx) \|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sen}^2(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \left(1 - \cos(2nx) \right) dx$$

$$\operatorname{sen}^2(a) = \frac{1}{2} \left(1 - \cos(2a) \right)$$

$$= \frac{1}{2\pi} \left[2\pi - \cancel{\frac{\operatorname{sen}(2nx)}{2n}}^{\,0} \Bigg|_{-\pi}^{\pi} \right] = 1$$

$$\| \cos(nx)\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(nx)\,dx = \frac{1}{2\pi}\left[2\pi + \overset{0}{\cancel{\frac{\operatorname{sen}(2nx)}{2n}}}\Bigg|_{-\pi}^{\pi}\right] = 1$$

$$\cos^2 a = \frac{1}{2}\left(1 + \cos(2a)\right)$$

1) $$\langle \operatorname{sen}(nx), \cos(mx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{\operatorname{sen}(nx)}^{\text{impar}}\,\underbrace{\cos(nx)}_{\text{par}}}_{\text{impar}}\,dx = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \operatorname{sen}(nx)\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\frac{1}{\sqrt{2}}\operatorname{sen}(nx)}_{\text{impar}}\,dx = 0$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos(nx)\right\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{\sqrt{2}}\cos(nx)\,dx = \frac{1}{\sqrt{2}\pi}\,\overset{0}{\cancel{\frac{\operatorname{sen}(nx)}{n}}}\Bigg|_{-\pi}^{\pi} = 0$$

$$\langle \operatorname{sen}(nx), \operatorname{sen}(mx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sen}(nx)\operatorname{sen}(mx)\, dx$$

$n \neq m$

$$\operatorname{sen}(a)\operatorname{sen}(b) = \frac{1}{2}\left[\cos(b-a) - \cos(b+a)\right]$$

$n \neq m$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((m-n)x) - \cos((n+m)x)\, dx = 0$$

$n \neq m$

$$\langle \cos(nx), \cos(mx) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n-m)x) + \cos((n+m)x)\, dx = 0$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(b-a) + \cos(b+a)\right]$$

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La serie de Fourier parcial de $f \in V$ es la proyección ortogonal de f sobre el espacio $\left\langle \left\{ \frac{1}{\sqrt{2}}, \operatorname{sen}(kx), \cos(kx) \right\}_{k=1}^{n} \right\rangle$

$$S_n(f) = \overbrace{\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}}}^{a_0/2} + \overbrace{\langle f, \operatorname{sen}(x) \rangle}^{b_1} \operatorname{sen}(x) + \overbrace{\langle f, \cos(x) \rangle}^{a_1} \cos(x)$$

$$+ \dots + \overbrace{\langle f, \operatorname{sen}(nx) \rangle}^{b_n} \operatorname{sen}(nx) + \overbrace{\langle f, \cos(nx) \rangle}^{a_n} \cos(nx)$$

$$\left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} = \frac{\overbrace{\langle f, 1 \rangle}^{a_0}}{2} = \frac{a_0}{2} \qquad a_0 = \langle f, 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$\boxed{S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \operatorname{sen}(kx)}$$

Resumen:

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(kx)\,dx \end{aligned}\right\}\ \text{Coeficientes de Fourier de } f.$$

La serie de Fourier:

$$S_\infty(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\,\mathrm{sen}(kx)$$

Preguntas:

- ¿$S_\infty(f)$ converge puntualmente? ← (Teo. de Dini)
- ¿$S_\infty(f)$ converge uniformemente? ← (Mayorante)
- ¿$S_\infty(f)$ converge en la norma de V? ← (completo)

__Propiedad:__ $$S_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \operatorname{sen}(kx)$$

La norma (en V) de $S_n(f)$ es:

$$\| S_n(f) \|^2 = \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2$$

__DEM:__ (Pitágoras)

$$\| S_n(f) \|^2 = \langle S_n(f), S_n(f) \rangle = \overbrace{\left\langle \frac{a_0}{2}, \frac{a_0}{2} \right\rangle}^{\frac{a_0^2}{4} \left(\frac{1}{\pi}\int_{-\pi}^{\pi} dx\right) = 2} + \sum_{k=1}^{n} \overbrace{\langle a_k \cos(kx), a_k \cos(kx) \rangle}^{a_k^2 \overbrace{\| \cos(kx) \|^2}^{1}}$$

$$+ \sum_{k=1}^{n} \underbrace{\langle b_k \operatorname{sen}(kx), b_k \operatorname{sen}(kx) \rangle}_{b_k^2 \underbrace{\| \operatorname{sen}(kx) \|^2}_{1}}$$

$$= \frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2$$

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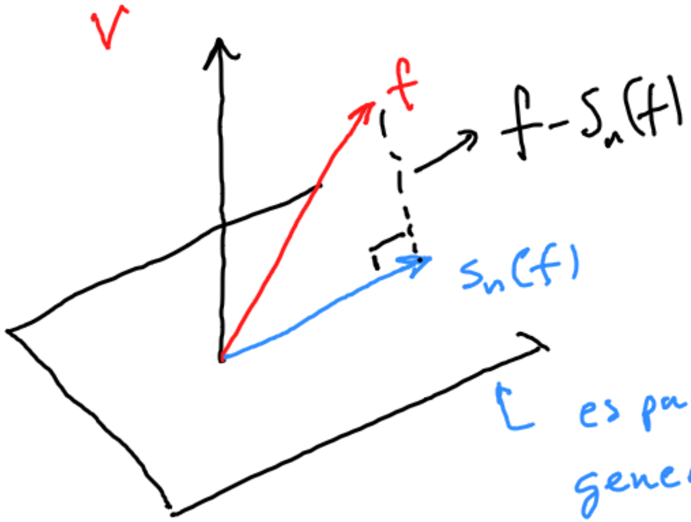


[espacio generado por primeros n cosenos y senos.

Prop: 1) $\|S_n(f)\| \leq \|f\|$

2) $\|f\|^2 = \|f - S_n(f)\|^2 + \|S_n(f)\|^2$

De 1) obtenemos:

DESIGUALDAD DE BESSEL:

$$\underbrace{\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2}_{\|S_n(f)\|^2} \leq \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx}_{\|f\|^2}$$

Observar:

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 < +\infty.$$

TEOREMA 0.2 : El conjunto S es completo :

$$\forall f \in V \text{ se tiene } \|f - S_n(f)\| \xrightarrow[n\to\infty]{} 0 .$$

DEM: Admitimos. $\quad (\text{en } V,\ S_\infty(f) = f)$

"el área de la recta es cero".

De 2) y el TEOREMA 0.2 obtenemos

IGUALDAD DE PARSEVAL : $\forall f \in V$: $\underbrace{\frac{a_0^2}{2} + \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2}_{\|S_\infty(f)\|^2} = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx}_{\|f\|^2}$

$$\|f\|^2 = \|f - S_n(f)\|^2 + \|S_n(f)\|^2$$

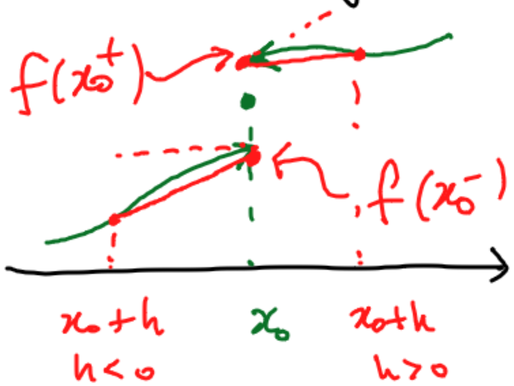
$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\|f\|^2 = 0 + \|S_\infty(f)\|^2$$

NOTACIÓN: Si $f \in V$ y $x_0 \in \mathbb{R}$:

$$f(x_0^+) = \lim_{x \to x_0^+} f(x)$$

$$f(x_0^-) = \lim_{x \to x_0^-} f(x)$$



TEOREMA DE DINI (Convergencia puntual de la serie de Fourier)

Sea $f \in V$ y tal que existen las derivadas laterales:

$$\forall x_0 \in \mathbb{R}: \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0^-)}{h} \text{ y } \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0^+)}{h} \text{ existen}$$

Entonces $S_\infty(f)(x_0) = \lim_{n \to \infty} S_n(f)(x_0) = \dfrac{f(x_0^+) + f(x_0^-)}{2}$

COROLARIO: Si f es continua entonces

(DINI)

$$S_\infty(f)(x) = f(x) \quad \forall x \in \mathbb{R}$$

(convergencia puntual)

Ejemplos: parte 2.g de la ficha 4.2.

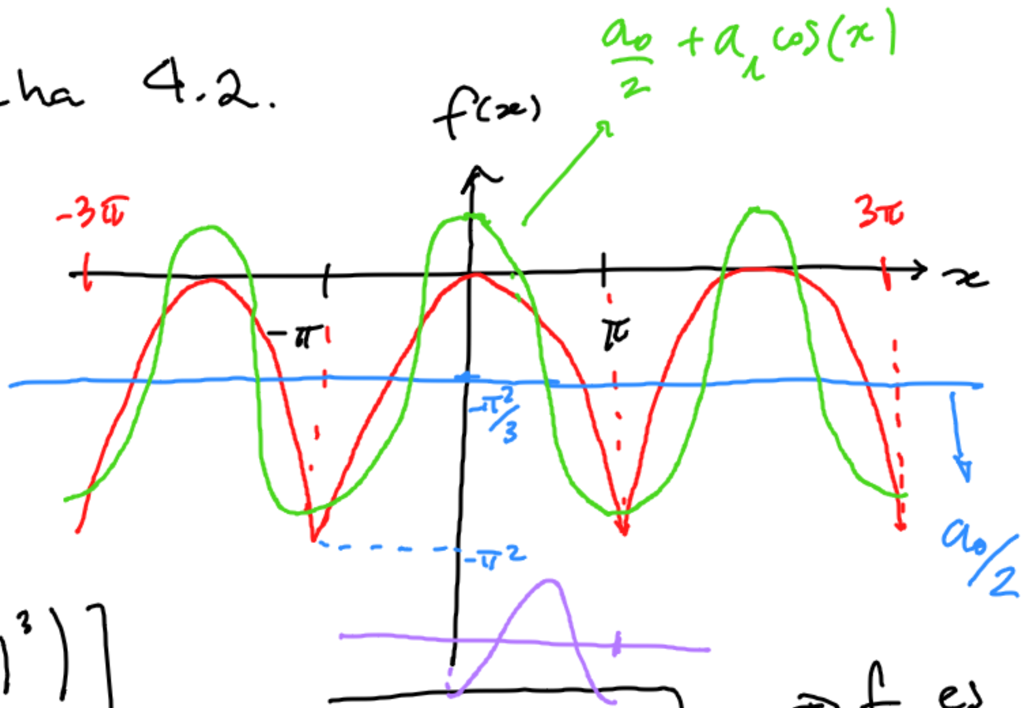
(i) $f(x) = -x^2$ en $[-\pi, \pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} -x^2\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{-\pi^3-\left(\cancel{+}(\cancel{+}\pi)^3\right)}{3}\right]$$

$$= \frac{-2\pi^2}{3}$$

$$\boxed{a_0/2 = \frac{-\pi^2}{3}.}$$



$$\boxed{b_k = 0 \quad \forall k} \rightarrow f \text{ es par}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\underbrace{(-x^2)}_{\text{par}}\,\underbrace{\text{sen}(kx)}_{\text{impar}}}^{\text{impar}}\,dx = 0$$

$$\frac{-\pi^2}{3} + 4\cos(x)$$

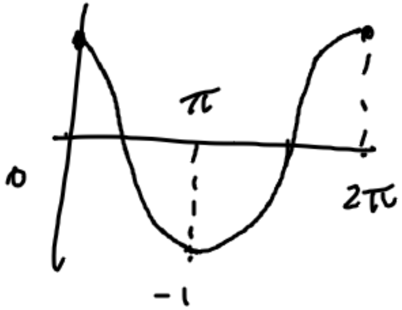
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{(-x^2)}_{\substack{\mu \\ D}} \underbrace{\cos(kx)}_{\substack{v' \\ I}}\, dx = \cancelto{0}{\frac{1}{\pi}(-x^2)\frac{\operatorname{sen}(kx)}{k}\Bigg|_{-\pi}^{\pi}}$$

$$+ \frac{2}{k\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\mu'}{x}}_{D}\, \underbrace{\overset{v}{\operatorname{sen}(kx)}}_{I}\, dx$$

$a_1 = 4$

$a_2 = -1$

$(-1)^k$



$$= \frac{2}{k\pi}\left[-x\frac{\cos(kx)}{k}\Bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(kx)}{k}\, dx \right]$$

$$= \frac{2}{k^2\pi}\left[-\pi(-1)^k - \pi(-1)^k + \cancelto{0}{\frac{\operatorname{sen}(kx)}{k}\Bigg|_{-\pi}^{\pi}} \right] = \frac{4(-1)^{k+1}}{k^2}$$

$$\boxed{a_k = \frac{4(-1)^{k+1}}{k^2}}$$