Lema de Gronwall y aplicaciones:

Una buena forma de interpretar el Lema de Gronwall es como un resultado de comparación de soluciones.

Lema de Gronwall: Sean $u, b : [T_1, T_2] \to \mathbb{R}$ con $b \geq 0$ continuas y $\alpha > 0$ constante tales que

$$\boxed{u(t) \leq \alpha + \int_{T_1}^{t} b(s)\,u(s)\,ds} \quad (*) \quad \forall\, t \in [T_1, T_2]$$

Comparar con:

$\dot{u} \leq bu$

$\dot{y} = by$

Entonces $u(t) \leq \underbrace{\alpha\, e^{\int_{T_1}^{t} b(s)\,ds}}$

Dem: Supongamos que $y(t)$ satisface la igualdad en (*):

$$y(t) = \alpha + \underbrace{\int_{T_1}^{t} b(s)\,y(s)\,ds}_{I(t)} \qquad I(T_1) = 0$$

$$\dot{I}(t) = b(t)\,y(t)$$

$b > 0$ $\qquad y(t) = \dfrac{\dot{I}(t)}{b(t)} \Rightarrow \dfrac{\dot{I}(t)}{b(t)} = \alpha + I(t)$

$$\begin{cases} \dot{I} - bI = \alpha b \\ I(T_1) = 0 \end{cases} \qquad B(t) = \int_{T_1}^{t} b(s)\,ds \qquad \dot{B} = b$$

$$\hookrightarrow e^{-B}\dot{I} - b e^{-B} I = \alpha b e^{-B}$$

$$\int_{T_1}^{t} \left(e^{-B} I \right)' ds = \int_{T_1}^{t} \left(-\alpha e^{-B} \right)' ds$$

$$e^{-B(t)} I(t) = -\alpha e^{-B(t)} + \alpha \underbrace{e^{\overbrace{-B(T_1)}^{0}}}_{1}$$

$$\boxed{I(t) = -\alpha + \alpha e^{B(t)}}$$

$$y(t) = \alpha + I(t) \Longrightarrow \boxed{y(t) = \alpha e^{B(t)}}$$

$$y(t) = \alpha e^{\int_{T_1}^{t} b(s)ds}$$

Recordar que queremos probar que $u(t) \leq y(t)$

$$u(t) \leq \alpha + \underbrace{\int_{T_1}^{t} b(s)\, u(s)\, ds}_{J(t)} \qquad u(t) \leq \alpha + J(t)$$

$$\dot{J}(t) = b(t)\, u(t) \qquad u(t) = \frac{\dot{J}(t)}{b(t)}$$

$$\frac{\dot{J}(t)}{b(t)} \leq \alpha + J(t)$$

$b \geq 0$

$$\dot{J}(t) - b(t)\, J(t) \leq \alpha\, b(t)$$

$$e^{-B(t)}\, \dot{J}(t) - b(t)\, e^{-B(t)}\, J(t) \leq \alpha\, b(t)\, e^{-B(t)}$$

$$e^{-B(t)} J(t) \leq -\alpha e^{-B(t)} + \alpha$$

$$J(t) \leq -\alpha + \alpha e^{B(t)}$$

$$u(t) \leq \alpha + J(t) \leq \alpha - \alpha + \alpha e^{B(t)} = \alpha e^{B(t)}$$

$$\boxed{u(t) \leq \alpha e^{\int_{T_1}^{t} b(s) ds}}$$

$\#$

Intervalos maximales de ecuaciones lineales: (Teo. 0.9)

Sean $A : \overset{(a,b)}{\mathbb{R}} \to M_{n\times n}(\mathbb{R})$ y $c : \overset{(a,b)}{\mathbb{R}} \to \mathbb{R}^n$ continuas

$A(t)$ matriz $\qquad$ $c(t)$ vector

Las soluciones maximales de $\dot{x} = A(t)x + c(t)$

están definidas en todo $\underline{\underline{\mathbb{R}}}$. (a,b)

Dem: $\varphi : (a,b) \to \mathbb{R}^n$ solución $\quad \varphi(t_0) = x_0$

$$\dot{\varphi}(t) = A(t)\varphi(t) + c(t) \qquad \forall t \in (a,b)$$

$$\varphi(t) - \underbrace{\varphi(t_0)}_{x_0} = \int_{t_0}^{t} A(s)\varphi(s) + c(s)\, ds$$

$$\Rightarrow \quad \varphi(t) = x_0 + \int_{t_0}^{t} \left[A(s)\varphi(s) + c(s)\right] ds \qquad t \geq t_0$$

Tomando la norma:

$$\|\varphi(t)\| = \left\| x_0 + \int_{t_0}^{t} \left[A(s)\varphi(s) + c(s)\right] ds \right\|$$

$$\leq \|x_0\| + \int_{t_0}^{t} \underbrace{\|A(s)\varphi(s)\|}_{\substack{\wedge\!\!| \\ \|A(s)\|\,\|\varphi(s)\|}} ds + \int_{t_0}^{t} \|c(s)\| ds$$

$$\leq \|x_0\| + \int_{t_0}^{t} \|c(s)\| ds + \int_{t_0}^{t} \|A(s)\|\,\|\varphi(s)\| ds$$

Supongamos que $b < +\infty$.

$$\|\varphi(t)\| \leq \|x_0\| + \int_{t_0}^{b} \|c(s)\| ds + \int_{t_0}^{t} \|A(s)\| \|\varphi(s)\| ds$$

$A(s)$ y $c(s)$ son continuas están acotadas en $[t_0, b]$.

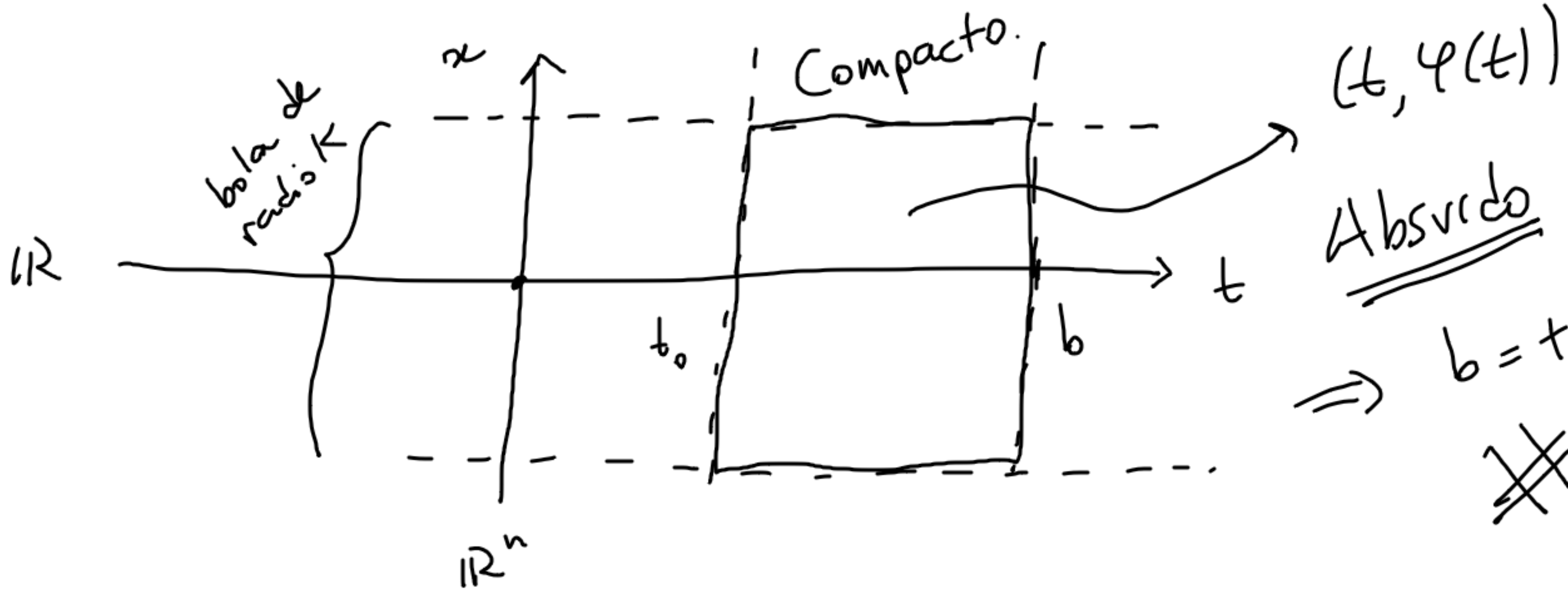
$\exists\ C \geq \|c(s)\|$, $M \geq \|A(s)\|$ $\forall\ s \in [t_0, b]$.

$$\rightarrow \underbrace{\|\varphi(t)\|}_{u(t)} \leq \overbrace{\|x_0\| + C(b - t_0)}^{\alpha} + \int_{t_0}^{t} \overbrace{M}^{b} \overbrace{\|\varphi(s)\|}^{u} ds.$$

$$u(t) \leq \alpha + \int_{t_0}^{t} b(s) u(s) ds$$

Por Gronwall $\Rightarrow$ $\underbrace{u(t)}_{\|\varphi(t)\|} \leq \alpha\, e^{\int_{t_0}^{t} b(s)\,ds} = \alpha\, e^{M(t-t_0)} \leq \alpha\, e^{M(b-t_0)}$

Conclusión: $\|\varphi(t)\| \leq \alpha\, e^{M(b-t_0)} = K$



Absurdo

$\Rightarrow b = +\infty$.

Estabilidad y teoremas de Liapunov (capítulo 6)

campo de velocidades

Consideremos la ecuación autónoma $\dot{x} = f(x)$ con

$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ loc. Lipschitziana ($\Rightarrow$ vale Picard)

↳ abierto y convexo

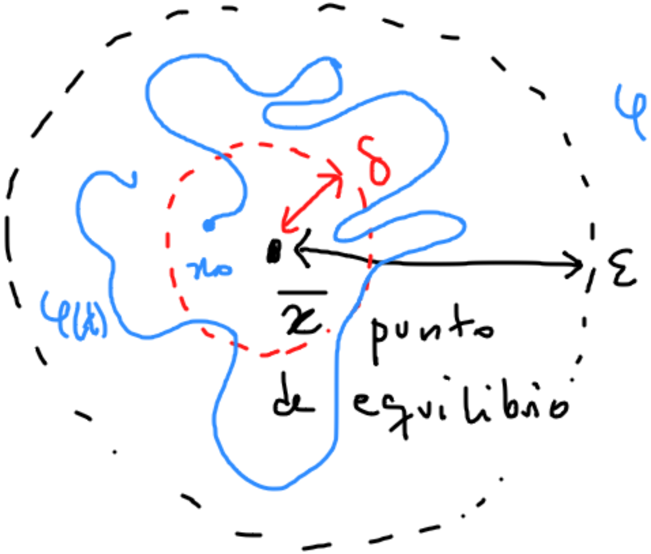
$\bar{x} \in \Omega$ es un punto de equilibrio si $f(\bar{x}) = 0$.

$\varphi(t) = \bar{x} \quad \forall t \in \mathbb{R}$ es una solución estacionaria.

<u>DEF</u>: $\bar{x}$ es estable si $\forall \varepsilon > 0$, $\exists \delta > 0$ t.q. si $\| x_0 - \bar{x} \| < \delta$

$\Rightarrow \quad \| \varphi(t) - \bar{x} \| < \varepsilon \quad \forall t \geq 0$ ($\varphi(t)$ es la solución con $\varphi(0) = x_0$)

Estable



$\varphi(t)$ no puede salirse de la bola de radio ε.

Asintóticamente estable: $\bar{x}$ es estable y ademas se cumple que $\exists\, \delta' > 0$ t.q. si $\|x_0 - \bar{x}\| < \delta' \Rightarrow \lim_{t \to +\infty} \varphi(t) = \bar{x}$

Inestable: $\bar{x}$ es inestable si no es estable.

Teoremas de Liapunov

$V : \Omega \longrightarrow \mathbb{R}$

$\varphi(t)$ solución

a $\dot{x} = f(x)$

$v(t) =$ el valor de V a lo largo de la solución

$= V(\varphi(t))$

producto escalar

$$\dot{v}(t) = \left(V(\varphi(t)) \right)^{\bullet} = \nabla_{\varphi(t)} V \cdot \dot{\varphi}(t) = \nabla_{\varphi(t)} V \cdot f(\varphi(t))$$

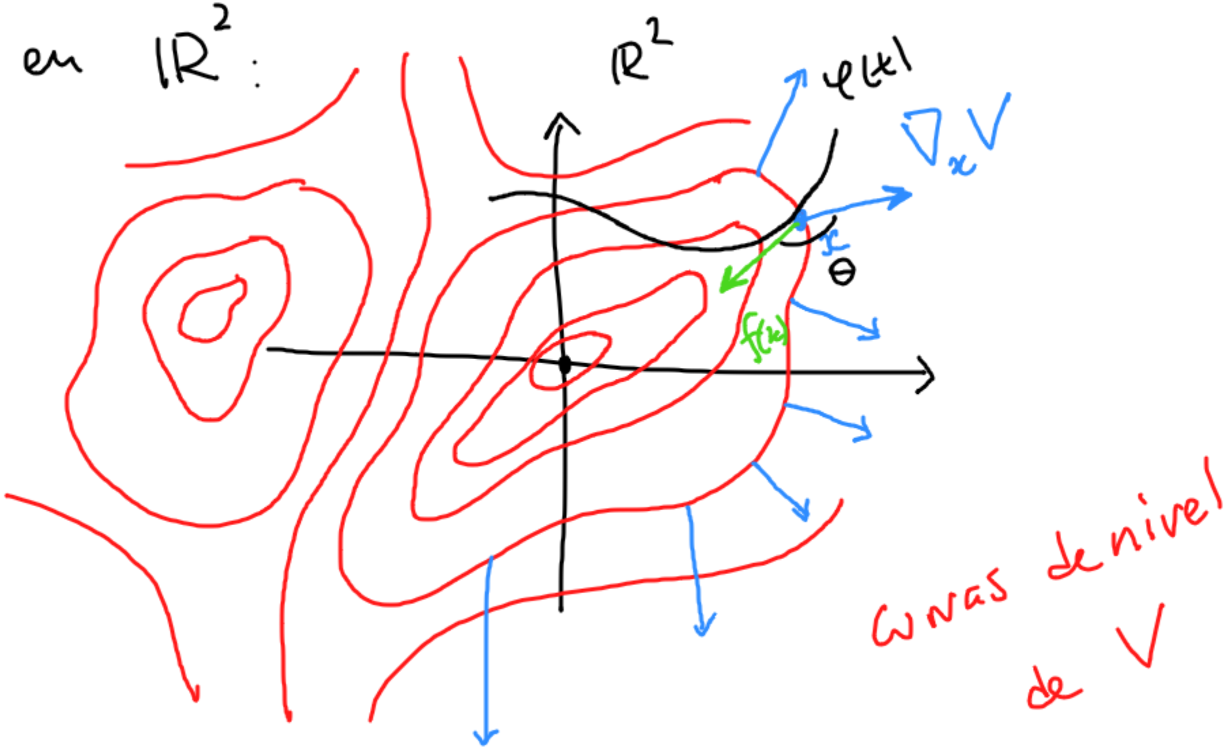
$$\boxed{\dot{V}(x) = \nabla_x V \cdot f(x)}$$

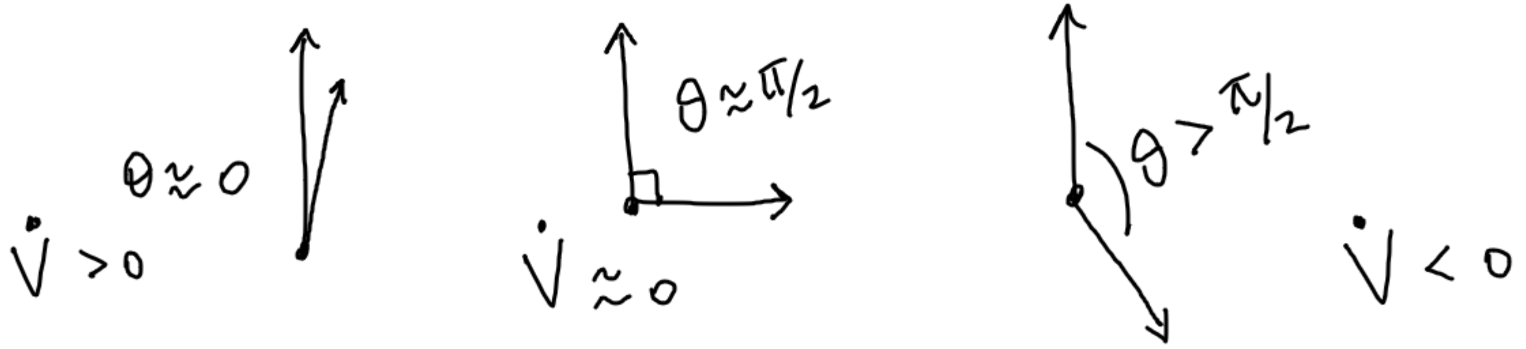
Interpretación geométrica en $\mathbb{R}^2$:

$$V : \mathbb{R}^2 \to \mathbb{R}$$

$$\dot{V}(x) = \nabla_x V \cdot f(x)$$

$$= \|\nabla_x V\| \, \|f(x)\| \cos\theta$$





DEF: Decimos que V es una pre-integral de la ecuación si $\dot{V}(x) = 0 \quad \forall x$.

Ejemplo: $m\ddot{x} + b\dot{x} + kx = 0$

$$\dot{x} = y$$

$$\dot{y} = -\frac{b}{m} y - \frac{k}{m} x$$

$$y = \dot{x}$$

$$\dot{y} = \ddot{x} = -\frac{b}{m} y - \frac{k}{m} x$$

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X}$$

$$= AX \qquad f(X) = AX \qquad V(X)$$

$$V(x,y) = \underbrace{\frac{1}{2} m y^2}_{\text{energía cinética}} + \underbrace{\frac{1}{2} k x^2}_{\text{energía potencial}}$$

$$\dot{V}(X) = \nabla_X V \cdot f(X)$$

$$V(X) = \frac{1}{2} m y^2 + \frac{1}{2} k x^2$$

$$\frac{\partial V}{\partial x} = kx \qquad \frac{\partial V}{\partial y} = my$$

$$\nabla_X V = \begin{pmatrix} kx \\ my \end{pmatrix}$$

$$f(X) = \begin{pmatrix} y \\ -\frac{b}{m} y - \frac{k}{m} x \end{pmatrix}$$

$b \geq 0$

$$\boxed{\dot{V}(X) = -by^2}$$

$$\dot{V}(x) = \cancel{kxy} - by^2 - \cancel{kxy}$$

Rozamiento $b > 0 \Rightarrow \dot{V} < 0$

No rotamiento $b = 0 \Rightarrow \dot{V} = 0$