Teorema de Picard y sus aplicaciones (soluciones maximales)

$$(*)_0 \begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

$f : \Omega \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ loc. Lipschitziana en la variable x y continua.

Ω abierto

(por ej. C^1)

Entonces:

1) $\exists\, \alpha > 0$ t.q. $(*_0)$ admite una única solución definida en por lo menos $(t_0-\alpha, t_0+\alpha)$ (α depende de (t_0,x_0))

2) $\exists\, r > 0$ t.q. si $\|(t_1,x_1)-(t_0,x_0)\| < r$

$$\Rightarrow (*_1) \begin{cases} \dot{x} = f(t,x) \\ x(t_1) = x_1 \end{cases}$$ admite solución única en $(t_1-\alpha, t_1+\alpha)$

Ejercicio 2 Ficha 3

a) $\begin{cases} \dot{x} = x \\ x(0) = 1 \end{cases}$ $\quad f(t,x) = x \quad \Omega = \mathbb{R}\times\mathbb{R}$ es C^1 en todo Ω

$(n=1) \Rightarrow$ loc. Lips y continua.

$$x(t) = e^t$$

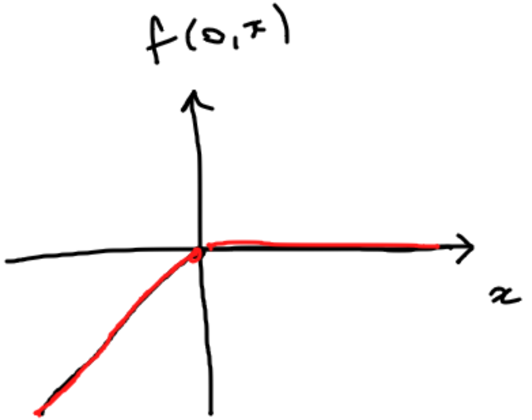
b) $\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$ $\quad f(t,x) = x^2 \quad \Omega = \mathbb{R}\times\mathbb{R}$ es C^1 en todo Ω

$\dfrac{\partial f}{\partial t} = 0 \quad \dfrac{\partial f}{\partial x} = 2x \quad \Rightarrow$ loc. Lips y continua.

c) $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ $\quad f(t,x) = \min\{|t|, x\} \quad \Omega = \mathbb{R}\times\mathbb{R}$

$$f(t,x) = \min\{|t|, x\}$$

Ejemplo: $t=0$ $\quad f(0,x) = \min\{0, x\}$

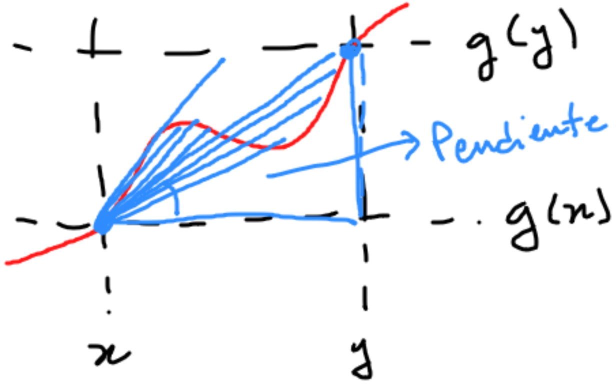


$f(t,x)$ NO es C^1.

¿ $\exists\, k>0$ t.q. $|f(t,y) - f(t,x)| \leq k|y-x|$ $\quad \forall\, t, y, x$.

$g(x) = f(t,x)$ fijando el t $\quad |g(y) - g(x)| \leq k|y-x|$

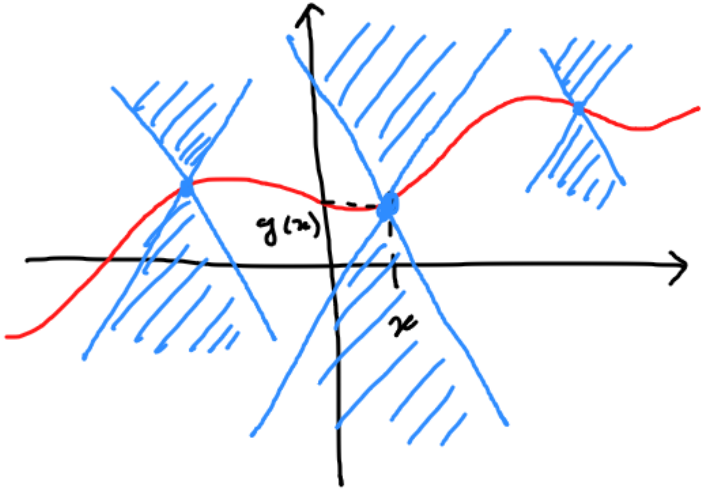
$$|\text{Pendiente}| = \left| \frac{g(y) - g(x)}{y - x} \right| \leq k$$



Lipschitziana $\Leftrightarrow$ Pendiente acotada

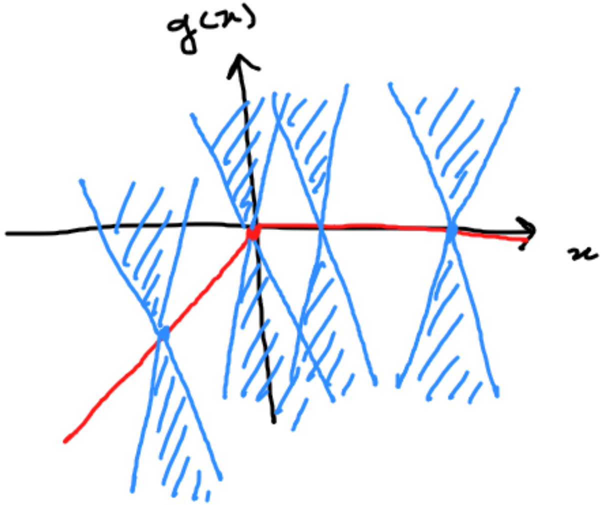
$$\left| \frac{g(y) - g(x)}{y - x} \right| \leq k \quad \Longleftrightarrow$$

$$\forall x, y$$

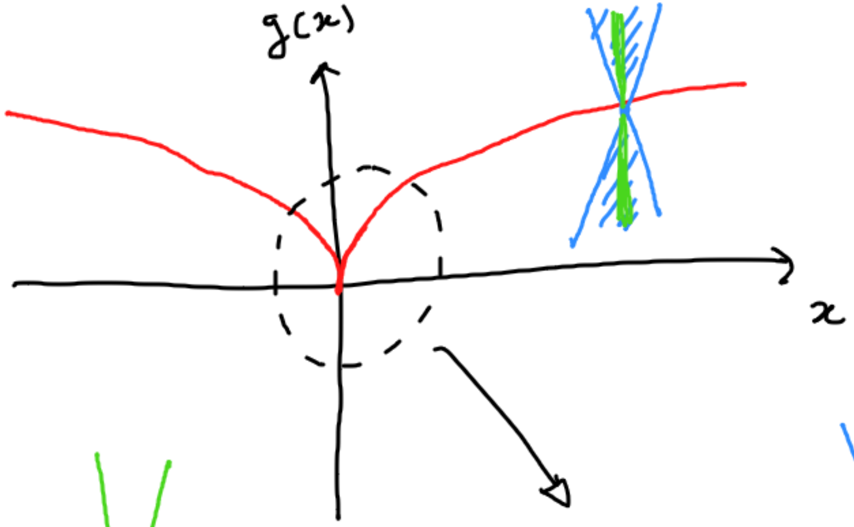


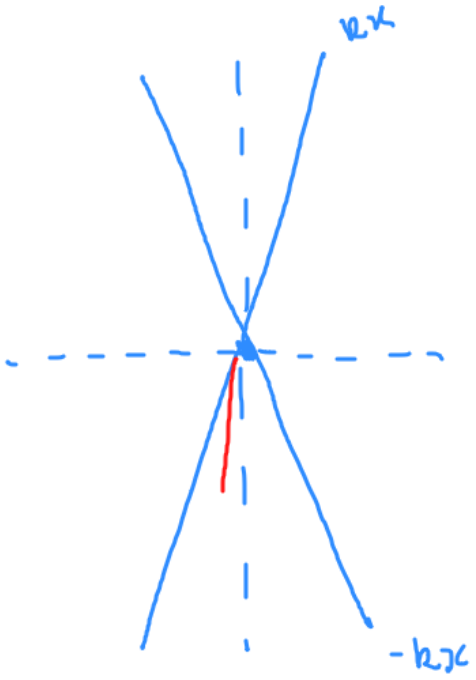
$$f(t,x) = \min\{|t|, x\}$$

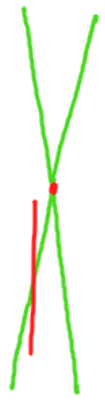
$$t=0 \qquad g(x) = \min\{0, x\}$$

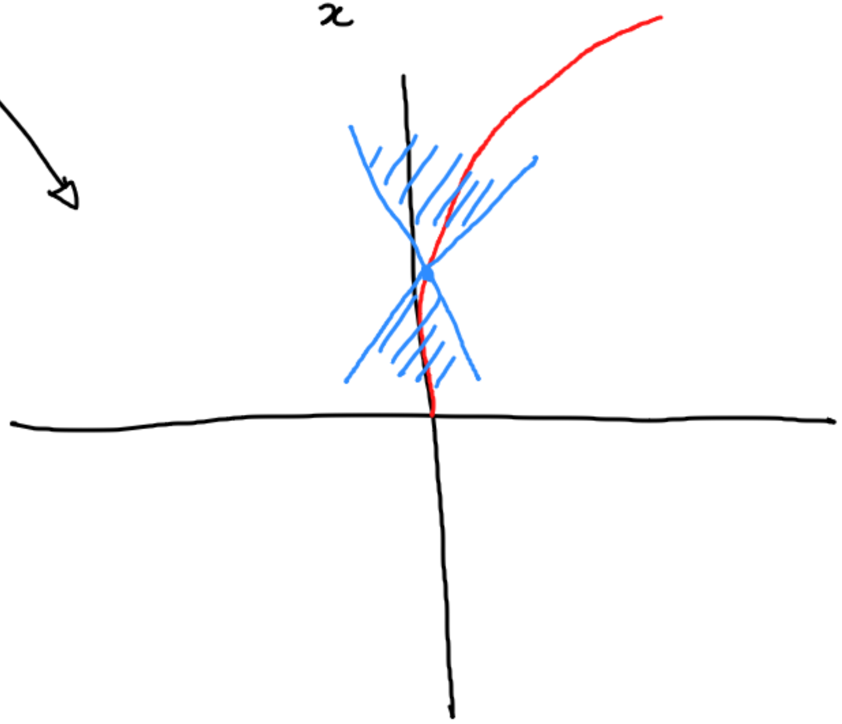


Por ejemplo: $g(x) = \sqrt{|x|}$ no es Lipschitziano (ni siquiera localmente)

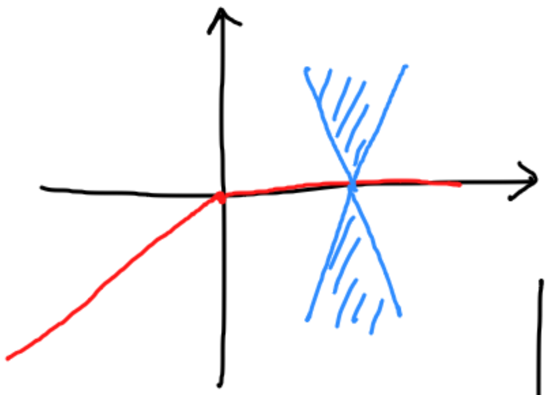








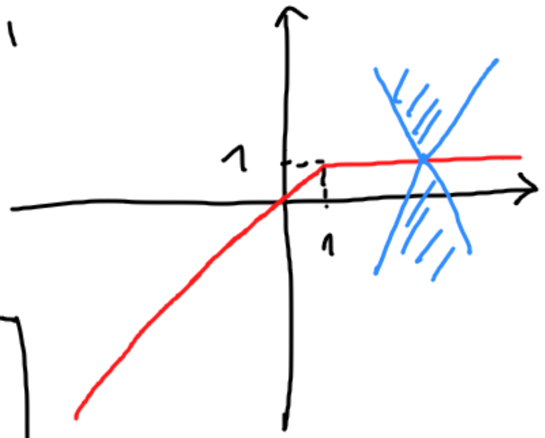
$t=0$



$g(x) = \min\{0, x\}$

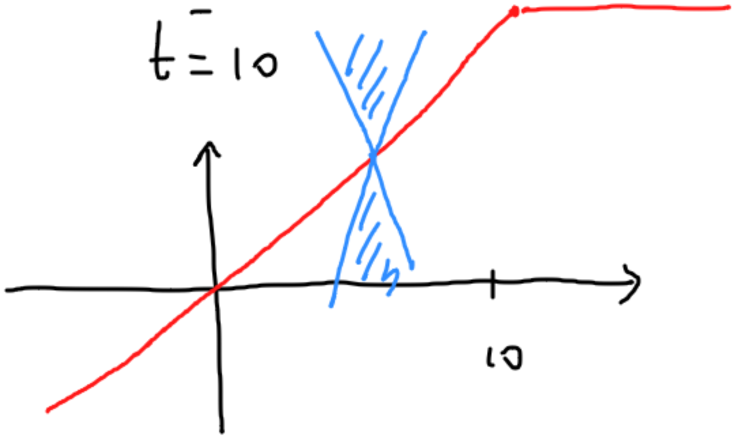
El mismo K sirve para todo t

$t=1$

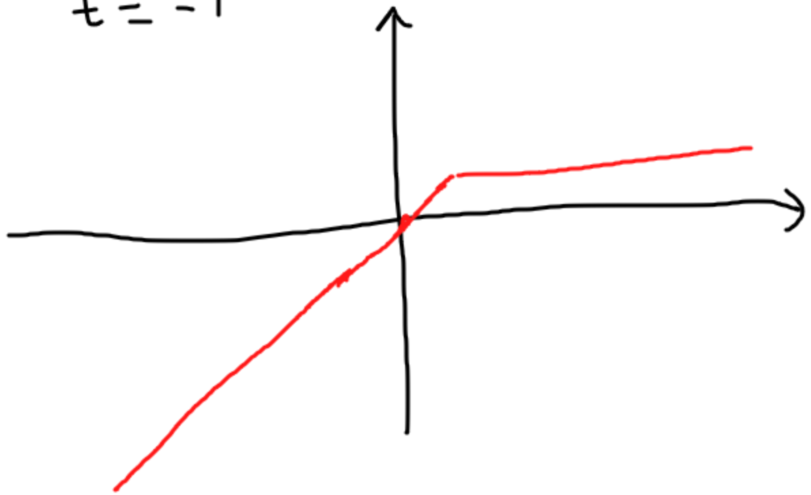


$g(x) = \min\{1, x\}$

$t=10$

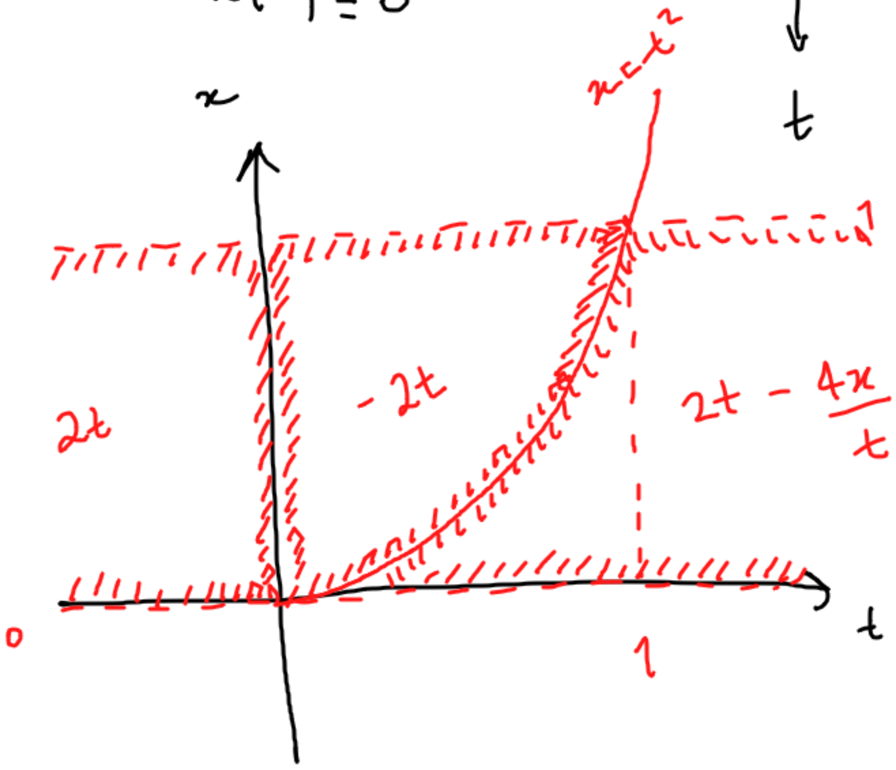


$t=-1$



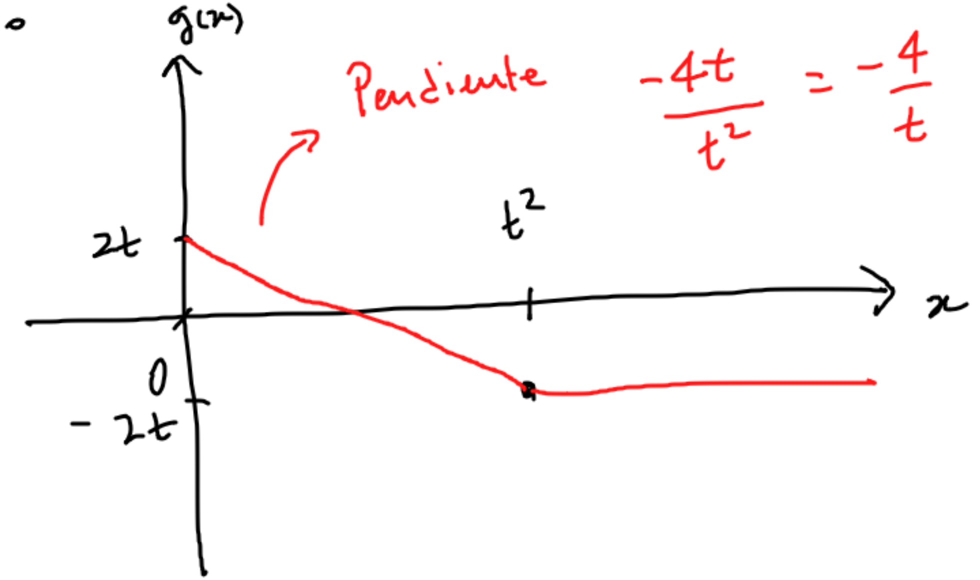
d) $\begin{cases} \dot{x} = f(t,x) \\ x(0) = 0 \end{cases}$ $\Omega = \underset{\downarrow}{\mathbb{R}} \times \underset{\downarrow}{(0,1)}$ (↓ t, ↓ x)

$$f(t,x) = \begin{cases} 2t & \text{si } t \le 0 \\ 2t - \dfrac{4x}{t} & \text{si } 0 < x < t^2,\ t > 0 \\ -2t & \text{si } t^2 \le x,\ t > 0 \end{cases}$$



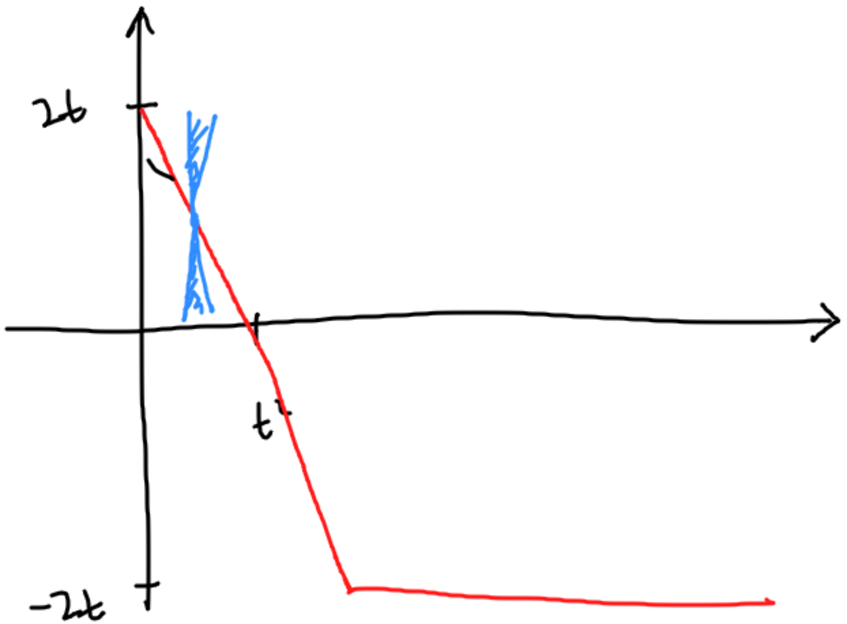
$g(x) = f(t,x)$ t fijo

$t>0$



$$\underline{\underline{t \approx 0}}$$

$$|\text{Pendiente}| = \left|-\frac{4}{t}\right| = \boxed{\frac{4}{t}} \approx +\infty$$



Para cualquier t

$g(x)$ es Lipschitz

$K = \frac{4}{t}$ No me sirve el mismo K para todo t cerca de $t=0$.

$\Rightarrow f(t,x)$ no es loc. Lips.

$n=2$ $\quad f(t,x) \quad (t,x) \in \mathbb{R}\times\mathbb{R}^2$

Fijamo t $\quad g(x) = f(t,x) \quad g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

gráfico $\quad (x, g(x)) \in \mathbb{R}^4$

Soluciones maximales:

- (I,φ) denota una solución de $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ definida en I

  $\varphi : I \to \mathbb{R}^n$ $\quad I$ = intervalo.

- (I,φ) es una extensión de (J,ψ) si $I \supseteq J$ y $\varphi|_J = \psi$

- (I,φ) es una solución maximal si su única extensión es ella misma.

  [f es loc. Lips y cont.]

Lema: (I_1,φ_1) y (I_2,φ_2) dos soluciones de $\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$

$$\Rightarrow \quad \varphi_1(t) = \varphi_2(t) \quad \forall\, t \in I_1 \cap I_2 .$$

Teorema 0.3 : $* \begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$ f loc. Lips y cont.

Entonces existe una única solución maximal.

Dem: $S = \{ (I, \varphi) \text{ solución de } * \}$

J = unión de todos los I que aparecen en S

$\Psi : J \to \mathbb{R}^n$ $\quad \Psi(t) = \varphi(t)$ si $t \in I$ $\quad (I,\varphi) \in S$

Posible problema: $t \in I_1$ $\quad (I_1, \varphi_1) \in S$

$t \in I_2$ $\quad (I_2, \varphi_2) \in S$

$\Psi(t) = \varphi_1(t)$ ó $\Psi(t) = \varphi_2(t)$? $\cancel{\ }$

No importa porque $\varphi_1(t) = \varphi_2(t)$.

Intervalo maximal = el intervalo I de definición de la solución maximal.

Se lo suele escribir $I(t_0, x_0)$ porque depende de las condiciones iniciales.

$$\dot{x} = \frac{dx}{dt} = x^2 - 1 = (x-1)(x+1)$$

<u>Ejemplo:</u> $\begin{cases} \dot{x} = x^2 - 1 \\ x(0) = x_0 \end{cases}$

$$\int_{x_0}^{x} \frac{dx}{(x-1)(x+1)} = \int_0^t dt = t$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\shortparallel$$

$A = 1/2 \quad B = -1/2$

$$\frac{1}{2}\int_{x_0}^{x} \frac{1}{x-1}\,dx - \frac{1}{2}\int_{x_0}^{x} \frac{1}{x+1}\,dx$$

$$= \frac{1}{2} \ln(x-1)\Big|_{x_0}^{x} - \frac{1}{2} \ln(x+1)\Big|_{x_0}^{x}$$

$$\frac{1}{2} \ln(x-1) \Big|_{x_0}^{x} - \frac{1}{2} \ln(x+1) \Big|_{x_0}^{x}$$

$$= \frac{1}{2} \ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2} \ln\left(\frac{x_0-1}{x_0+1}\right) = t$$

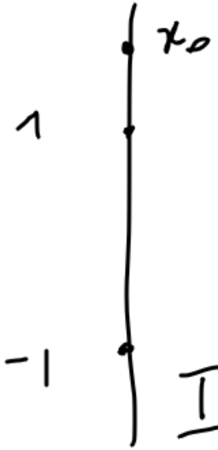
$$\ln\left(\frac{x-1}{x+1}\right) = \ln\left(\frac{x_0-1}{x_0+1}\right) + 2t$$

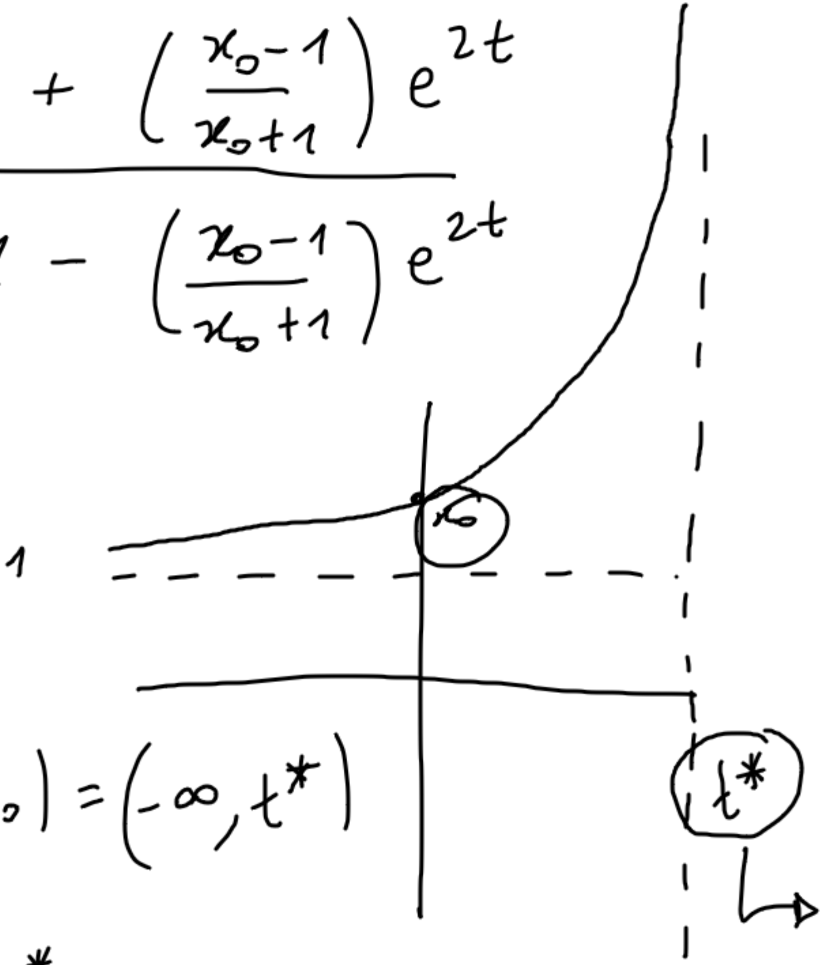
$$\frac{x-1}{x+1} = \left(\frac{x_0-1}{x_0+1}\right) e^{2t}$$

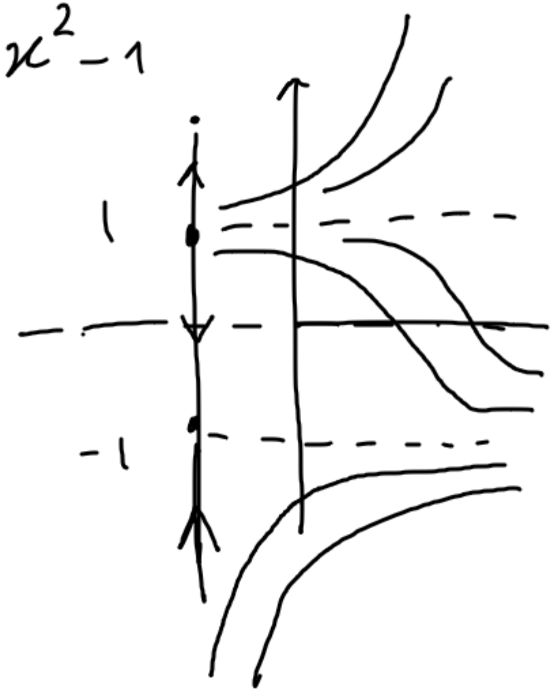
$$x - 1 = \left(\frac{x_0-1}{x_0+1}\right) e^{2t} (x+1)$$

$$\left(1 - \left(\frac{x_0-1}{x_0+1}\right) e^{2t}\right) x = 1 + \left(\frac{x_0-1}{x_0+1}\right) e^{2t}$$

$$x(t) = \frac{1 + \left(\frac{x_0 - 1}{x_0 + 1}\right) e^{2t}}{1 - \left(\frac{x_0 - 1}{x_0 + 1}\right) e^{2t}}$$







$$I(0, x_0) = (-\infty, t^*)$$

$$t^* = \frac{1}{2} \ln\left(\frac{x_0 - 1}{x_0 + 1}\right)$$

$$\left(\frac{x_0 - 1}{x_0 + 1}\right) e^{2t^*} = 1$$

$$\ln\left(\frac{x_0 - 1}{x_0 + 1}\right) + 2t^* = 0$$