Matriz fundamental:

DEF: (E) $\dot{X} = AX$ con $A \in M_{n\times n}(\mathbb{R})$ y $X \in \mathbb{R}^n$.

Consideremos la ecuación matricial (E*) $\begin{cases} \dot{M} = AM \\ M(0) = Id. \end{cases}$

La solución $M(t) \in M_{n\times n}$ es la matriz fundamental de (E).

EQUIV: $M = [X_1, \dots, X_n]$ X_i: la i-ésima columna de M.

La $X_i \in \mathbb{R}^n$ es solución de (E_i) $\begin{cases} \dot{X} = AX \\ X(0) = e_i \end{cases}$

con $\{e_1, \dots, e_n\}$ base canónica de $\mathbb{R}^n$.

CONS: $\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$ con $X_0 \in \mathbb{R}^n$ cualquiera $\Rightarrow X(t) = M(t)X_0$

Ejemplos:

(1) $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \Rightarrow M(t) = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}$

(2) $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow M(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

(3) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow M(t) = e^{at} \begin{pmatrix} \cos(bt) & \operatorname{sen}(bt) \\ -\operatorname{sen}(bt) & \cos(bt) \end{pmatrix}$

DEF: Exponencial de una matriz: $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

PROP: $(E)\ \dot{X} = AX \Rightarrow \boxed{M(t) = e^{At}}$

CONS: $(E_0) \begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \Rightarrow \boxed{X(t) = e^{At} X_0}$

Cálcular la exponencial de una matriz

PROP: $B = P^{-1} A P \Rightarrow e^{B} = P^{-1} e^{A} P$

PROP: $A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right) \Rightarrow e^{A} = \left(\begin{array}{c|c} e^{B} & 0 \\ \hline 0 & e^{C} \end{array}\right)$

PROP: Si A y B conmutan $AB = BA \Rightarrow e^{A+B} = e^{A} e^{B}$

Consecuencia $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$ $\qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} A^k \frac{1}{(n-k)!} B^{n-k} = \underbrace{\left(\sum_{n=0}^{\infty} \frac{1}{n!} A^n\right)\left(\sum_{m=0}^{\infty} \frac{1}{m!} B^m\right)}_{e^{A} e^{B}}$$

1) Matriz diagonal $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$

$$e^{At} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 t & & 0 \\ & \ddots & \\ 0 & & \lambda_n t \end{pmatrix} + \begin{pmatrix} \frac{(\lambda_1 t)^2}{2!} & & 0 \\ & \ddots & \\ 0 & & \frac{(\lambda_n t)^2}{n!} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+\lambda_1 t + (\lambda_1 t)^2/2! + \cdots & & 0 \\ & \ddots & \\ 0 & & 1+(\lambda_n t)+(\lambda_n t)^2/2! + \cdots \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

2) Matriz de Jordan $A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{J}$

$A = D + J$

Observar que $DJ = JD \Rightarrow e^{A} = e^{D+J} = e^{D} e^{J}$

$$e^{At} = e^{(D+J)t} = e^{Dt+Jt} = e^{Dt}\,e^{Jt} = e^{\lambda t}\,e^{Jt}$$

$$D = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} = \lambda\, Id \Rightarrow e^{Dt} = e^{\lambda t}\, Id$$

<u>$e^{Jt} = ?$</u> :

$n=2$ $\quad J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad J^2 = 0$

$n=3$ $\quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow J^3 = 0$

En general $J^n = 0$

$$e^{Jt} = Id + Jt + \frac{(Jt)^2}{2!} + \cdots + \frac{(Jt)^{n-1}}{(n-1)!} + \underbrace{----}_{0}$$

porque $J^n = 0$

$n = 4$

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^4 = 0$$

$$e^{Jt} = Id + \begin{pmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & t^2/2! & 0 \\ 0 & 0 & 0 & t^2/2! \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & t^3/3! \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t & t^2/2 & t^3/3! \\ & 1 & t & t^2/2 \\ & & 1 & t \\ 0 & & & 1 \end{pmatrix}$$

En general: $$e^{Jt} = \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & t^2/2! \\ & & & \ddots & t \\ & 0 & & & 1 \end{pmatrix}$$

En resumen: $$\boxed{e^{At} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & t^2/2! \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}}$$

En el caso $n=2$: $$e^{At} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Ejercicio 13 iii) $\quad A = \begin{pmatrix} 2 & 3 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

Valores propios : $\det(A-\lambda I) = \begin{pmatrix} 2-\lambda & 3 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{pmatrix}$

$$= (2-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(1+\lambda)^2$$

doble

$\boxed{2 \text{ y } -1}$

Espacios propios: $\lambda = 2$: $\begin{pmatrix} 0 & 3 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix} \Rightarrow \boxed{v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$y = z = 0$

$\lambda = -1$ $\quad \begin{pmatrix} 3 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+3y+\cancel{2z} \\ z = 0 \\ 0 \end{pmatrix} \Rightarrow x = -y \Rightarrow \{(x,-x,0) : x \in \mathbb{R}\}$

$\Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$(A - \lambda I)\, v_3 = v_2$$

$\lambda = -1$

$$\underbrace{\begin{pmatrix} 3 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3x + 3y + 2z \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow$$

$z = -1$

$3x + 3y = 3$

$x + y = 1$

$y = 1 - x$

$$v_3 = (x,\ 1 - x,\ -1)$$

$$\boxed{v_3 = (1, 0, -1)} \qquad \boxed{P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

$$P^{-1} A P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow A = P \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} P^{-1}$$

$$e^{At} = P \, \underbrace{e^{\overset{B}{\left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right)}_{C} t}} \, P^{-1} = P \left(\begin{array}{c|c} e^{Bt} & 0 \\ \hline 0 & e^{Ct} \end{array}\right) P^{-1}$$

$$\left.\begin{aligned} e^{Bt} &= e^{2t} \\ e^{Ct} &= e^{-t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \end{aligned}\right\} \quad e^{At} = P \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & e^{-t} t \\ 0 & 0 & e^{-t} \end{pmatrix} P^{-1}$$

Ejemplo: Resorte (forzado) con rozamiento

$$m\ddot{x} + b\dot{x} + kx = F_{ext}(t)$$

$F_{ext} = 0$:

$y = \dot{x}$ $\qquad \dot{y} = \ddot{x} = -b/m\, \dot{x} - k/m\, x = -b/m\, y - k/m\, x$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -b/m\, y - k/m\, x \end{cases} \iff \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{X} = AX$$

$$X(t) = e^{At} X_0 \qquad \omega_0^2 = k/m \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -b/m \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -w_0^2 & -\gamma \end{pmatrix}$$

$$\gamma = b/m$$

<u>Valores propios:</u>

$$\left| \begin{pmatrix} -\lambda & 1 \\ -w_0^2 & -\gamma-\lambda \end{pmatrix} \right|$$

$$= -\lambda(-\gamma-\lambda) + w_0^2$$

$$= \lambda(\lambda+\gamma) + w_0^2$$

$$p(\lambda) = \lambda^2 + \gamma\lambda + w_0^2$$

$$\boxed{\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4w_0^2}}{2}}$$

(1) $\to \gamma > 2w_0$ A diagonal

(2) $\to \gamma = 2w_0$?

(3) $\to \gamma < 2w_0$ complejos

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4w_0^2}}{2}$$

Caso (1): $\gamma > 2w_0$

En la base de vectores propios

$$e^{At} = \begin{pmatrix} e^{t\left(\frac{-\gamma + \sqrt{\gamma^2 - 4w_0^2}}{2}\right)} & 0 \\ 0 & e^{t\left(\frac{-\gamma - \sqrt{\gamma^2 - 4w_0^2}}{2}\right)} \end{pmatrix}$$

$\gamma/2 = w_0$

caso (2): $\gamma = 2w_0$

$\lambda = -\gamma/2$

$$A + \gamma/_2 Id = \begin{pmatrix} \gamma/2 & 1 \\ -w_0^2 & -\gamma/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ -w_0^2 & -\gamma \end{pmatrix}$$

$$-w_0 \begin{pmatrix} w_0 x + y \\ -w_0^2 x - w_0 y \end{pmatrix} = \begin{pmatrix} \frac{\gamma}{2} x + y \\ -w_0^2 x - \frac{\gamma}{2} y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$y = -w_0 x$

Espacio propio: $\{(x, -w_0 x) : x \in \mathbb{R}\}$ dim = 1

$$v_1 = \begin{pmatrix} 1 \\ -w_0 \end{pmatrix}$$

$$A + \frac{\gamma}{2} Id = \begin{pmatrix} w_0 & 1 \\ -w_0^2 & -w_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -w_0 \end{pmatrix}$$

$$= \begin{pmatrix} w_0 x + y \\ -w_0^2 x - w_0 y \end{pmatrix} = \begin{pmatrix} 1 \\ -w_0 \end{pmatrix} \Rightarrow y = 1 - w_0 x$$

$$v_2 = (x, 1 - w_0 x) \Rightarrow v_2 = (0, 1)$$

Base $\left\{ \begin{pmatrix} 1 \\ -w_0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ $\quad P = \begin{pmatrix} 1 & 0 \\ -w_0 & 1 \end{pmatrix}$ $\quad P^{-1} A P = \begin{pmatrix} -w_0 & 1 \\ 0 & -w_0 \end{pmatrix}$

$$B = P^{-1} A P = \begin{pmatrix} -\omega_0 & 1 \\ 0 & -\omega_0 \end{pmatrix} \qquad e^{Bt} = e^{-\omega_0 t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ -\omega_0 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ -\omega_0 & 1 \end{pmatrix} e^{-\omega_0 t} \underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \omega_0 & 1 \end{pmatrix}}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ \omega_0 & 1 \end{pmatrix}$$

$$= e^{-\omega_0 t} \begin{pmatrix} 1 & 0 \\ -\omega_0 & 1 \end{pmatrix} \begin{pmatrix} 1 + t\omega_0 & t \\ \omega_0 & 1 \end{pmatrix}$$

$x_0, v_0 = y_0$

$$e^{At} = e^{-\omega_0 t} \begin{pmatrix} 1 + t\omega_0 & t \\ -\omega_0^2 t & -\omega_0 t + 1 \end{pmatrix}$$

$$e^{At}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = e^{-\omega_0 t}\begin{pmatrix} 1+t\omega_0 & t \\ -\omega_0^2 t & -\omega_0 t+1 \end{pmatrix}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

$$\boxed{x(t) = e^{-\omega_0 t}\left((1+t\omega_0)\, x_0 + t\, v_0 \right)}$$

Estabilidad: $\dot{X} = AX$

DEF: Estable: X sol. con $X(0) = X_0$
(a futura) Y sol con $Y(0) = Y_0$

$X(t)$ es estable si $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ t.q.
si $\|Y_0 - X_0\| < \delta \Rightarrow \|Y(t) - X(t)\| < \varepsilon \quad \forall\, t \geq 0$

$$\left.\begin{array}{l} Y(t) = e^{At} Y_0 \\ X(t) = e^{At} X_0 \end{array}\right\} \Rightarrow \|Y(t) - X(t)\| = \| e^{At} (\underbrace{Y_0 - X_0}_{Z}) \|$$

Estable $\Leftrightarrow$ si $\|Z\| < \delta \Rightarrow \| e^{At} Z \| < \varepsilon \quad \forall\, t$

Ejemplo: $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\quad e^{At} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}$

- $z = \begin{pmatrix} \delta/2 \\ 0 \end{pmatrix}$ $\quad \|z\| = \delta/2 < \delta \quad \|e^{At} z\| = \delta/2 \, e^{\alpha t} < \varepsilon\,?$
- $z = \begin{pmatrix} 0 \\ \delta/2 \end{pmatrix}$ $\quad \|z\| = \delta/2 < \delta \quad \|e^{At} z\| = \delta/2 \, e^{\beta t} < \varepsilon\,?$

es estable sólo si α y β son ≤ 0.

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{cases} \text{Estable si } \alpha \leq 0 \text{ y } \beta \leq 0 \begin{cases} \text{Atractor } \boxed{\text{Asint. Est. } \alpha < 0 \text{ y } \beta < 0} \\ \text{No} \end{cases} \\ \text{Inestable si } \alpha > 0 \text{ ó } \beta > 0 \end{cases}$$

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad e^{At} = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$\lambda < 0 \Rightarrow$ Asint. Est.

$\lambda = 0$ $\quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad e^{At} z = \begin{pmatrix} t\,\delta/2 \\ \delta/2 \end{pmatrix}$

Inestable. $\quad z = \begin{pmatrix} 0 \\ \delta/2 \end{pmatrix}$

$\lambda > 0$ $\quad \| e^{At} z \| = \delta/2 \, \underbrace{\sqrt{t^2+1}}_{\sim t} \longrightarrow +\infty$

Inestable

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \qquad e^{At} = e^{at} \begin{pmatrix} \cos(bt) & \operatorname{sen}(bt) \\ -\operatorname{sen}(bt) & \cos(bt) \end{pmatrix}$$

$a < 0$ Asint. Est.

$a = 0$ <u>Est.</u>

$a > 0$ Inestable.